VARIATIONAL PRINCIPLES AND APPROXIMATION OF DYNAMICAL INDICATORS FOR SYSTEMS WITH NONUNIFORMLY HYPERBOLIC BEHAVIOR

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Abstract. This note is concerned with approximation of dynamical indicators as pressures, Lyapunov exponents and dimension-like quantities, in systems with nonuniformly hyperbolic behavior. For this we let $P^*(\Phi) := \sup_\mu\{h(\mu) + \mu(\Phi)\}$ be a variational pressure defined over a suitable class of Borel measurable potentials and prove that, for regular nonuniformly hyperbolic systems, $P^*(\Phi) = \sup_\Omega P^*(f|\Omega, \Phi)$, supremum taken over the family of f-invariant uniformly hyperbolic basic sets. We then apply this variational principle to the approximation of dynamical indicators by horseshoes, generalizing results of Katok and Mendoza in [16].

1. Introduction

This paper is concerned with approximation of dynamical indicators –entropies, pressures, Lyapunov exponents and dimension-like characteristics– by sequences of hyperbolic Cantor sets. This type of questions has been considered previously by several authors. In 1977 R. Bowen proposed to look for large invariant subsets in the non-wandering set $\Omega(f)$ of a diffeomorphism on which the dynamics of f are simple to describe and raised the following:

Question: *Let small $\epsilon > 0$ be given. Does there exist a hyperbolic f-invariant subset $\Lambda_\epsilon \subseteq \Omega(f)$ such that $f \mid \Lambda_\epsilon$ is conjugated to a subshift of finite type and $h_{top}(f \mid \Lambda_\epsilon) > h_{top}(f) - \epsilon$?* [6, Chapter 10].

L.-S. Young answered this question affirmatively in 1981, proving that Axiom A diffeomorphisms and flows, piecewise monotonic maps of the interval, the Poincaré map of the Lorenz attractor and certain Abraham-Smale examples are limits of subshifts of finite type in the above sense. See [33].

Later, in 1984, A. Katok laid the foundations to study these questions in his seminal paper [14] about relations between entropy, periodic orbits and Lyapunov exponents of systems with nonuniformly hyperbolic behavior. The following proposition, proved in [16], gives a taste of this type of results: *let μ be an ergodic Borel probability with non zero Lyapunov exponents and $h(\mu) > 0$, then there exists a sequence of horseshoes Ω_n and ergodic measures μ_n supported on Ω_n such that $\mu_n \to \mu$ weakly and $h(\mu_n) \to h(\mu)$* (see). We refer the reader to [11], [12], [13], [18] [19], [30], [31] and [36] for some recent contributions to the subject.

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In this note we use ideas from non additive thermodynamic formalism and methods of [16] to approximate entropies, pressures, Lyapunov exponents, rates of escapes and certain dimension-like quantities, using a variational principle for systems with nonuniformly hyperbolic dynamics characterized by the abundance of hyperbolic Cantor sets.

1.1. **Variational principles.** Thermodynamic formalism is the primary source of variational principles in dynamical systems. Its main ingredientes are the *pressure* $P(\phi)$ of a continuous potential ϕ, a topological invariant of the dynamics introduced by D. Ruelle in [26] for a class of $\mathbb{Z}^n$ actions arising naturally in the formalism of equilibrium statistical physics. Later, in [32], P. Walters extended this notion for continuous maps f of compact metric space proving the following well known

Additive Variational Principle *Let f be a continuous transformation of a compact metric space and ϕ continuous. Then,*

$$P(\phi) = \sup_{\mu \in \mathcal{M}_f} \left\{ h(\mu) + \int \phi d\mu \right\}. \tag{1}$$

$h(\mu)$ is the *Kolmogorov-Sinai entropy* of an invariant Borel probability μ and supremum is taken over $\mathcal{M}_f$, the set of f-invariant Borel probabilities endowed with the weak topology. We call $E(\phi, \mu) := h(\mu) + \int \phi d\mu$ the *free energy.*

Walter's variational principle extends a similar result for entropies due to Dinaburg. See [20].

A Borel probability μ is called an *equilibrium state* if it is maximum of the free energy, i.e. $P(\phi) = h(\mu) + \int \phi d\mu$. Existence and uniqueness of equilibrium states is a central issue of the theory and depends on dynamical properties of the map f and the regularity of the potential ϕ.

A classical result due to Ruelle and Bowen claims that if f is a topologically transitive Anosov or Axiom A diffeomorphism and if ϕ is a Hölder continuous potential then there exists a unique equilibrium state $\mu = \mu_\phi$ which is moreover a *Gibbs measure*. See [7]. Measures of maximal entropy, absolutely continuous and Sinai-Ruelle-Bowen (SRB) measures and measures of maximal dimension of systems satisfying Axiom A are equilibrium states of suitable potentials. See [29].

Developing thermodynamics beyond the realm of uniformly hyperbolic dynamics faces several obstacles. For example one need to extend the notion of pressure for a larger class of measurable but not necessarily continuous potentials. For this a new concept of pressure is required and new methods are needed to establish the existence and uniqueness of equilibrium states originating several forms of non additive thermodynamic formalisms. See subsection 7.3.

Let us introduce some definitions before to make a formal statement.

Definition 1.1. We define the *variational pressure* of a Borel measurable potential Φ as

$$P^*(\Phi) := \sup \left\{ h(\mu) + \int \Phi d\mu : \mu \in \mathcal{M}_f \right\}, \tag{2}$$

where we require Φ to be integrable with respect to (w.r.t.) every f-invariant Borel probability $\mu \in \mathcal{M}_f$.

An issue will be to choose a domain to make this a well defined functional. Also we would like to see whether or not this is a topological invariant of the dynamics. We shall see below that both questions can be answered affirmatively for a class of admissible potentials $\mathcal{S}^+$ which contains all the continuous functions. In brief, $\Phi \in \mathcal{S}^+$ if either is continuous or is the rate of growing (12) of a sub(super) additive sequence ϕ_n of continuous functions. See section 3. The following is our

Main Theorem *Let f be a nonuniformly hyperbolic $C^{1+\alpha}$ diffeomorphism of a compact manifold and suppose that f leaves invariant at least a non atomic Borel probability then, for every admissible potential $\Phi \in \mathcal{S}^+$*

$$P^*(\Phi) = \sup\{ P^*(f|\Omega, \Phi) \,:\, \Omega \in \mathcal{H} \}, \tag{3}$$

where $\mathcal{H}$ is the family of f-invariant uniformly hyperbolic basic sets $\Omega \subset M$.

Remark 1.1. There are several examples of nonuniformly hyperbolic dynamics whose invariant measures are supported at finitely many periodic orbits. See [14, pp. 140] and [5]. Therefore existence of non atomic hyperbolic measure is a necessary condition for a nonuniformly hyperbolic system to be suitable for the present approach. We refer to these as *regular nonuniformly hyperbolic systems.* See section 2.

1.2. **Applications.** Variational equation (3) generalizes some of the results in [16] and provide new perspective on the subject focusing on variational principles to deal with this type of problems.

For instance, for given a continuous function we prove that

$$P(\phi) = \sup_{\Omega} P(f|\Omega, \phi),$$

supremum taken over the set of hyperbolic basic sets. See section 6 and compare [16]. In particular, there exists sequences of hyperbolic Cantor sets Ω_n such that $P(f|\Omega_n, \phi) \to P(\phi)$. We have similar results for topological entropy:

$$h(f) = \sup_{\Omega} h(f|\Omega),$$

from which we get sequences Ω_n such that $h(f|\Omega_n) \to h(f)$.

The next variational equation was first proved in [12] under a slightly different set of conditions and was our main motivation to prove (3):

$$\sup_{\mu \in \mathcal{M}_f} \left\{ h(\mu) + \int \phi^u d\mu \right\} = \sup_{\Omega \in \mathcal{H}} P(f|\Omega, \phi^u), \tag{4}$$

where the *unstable potential*,

$$\phi^u = -\log \mathrm{Jac}(Df|E^u), \tag{5}$$

is the induced volume deformation along unstable manifolds. By hyperbolic dynamics, $\phi^u \mid \Omega$ extends to a Hölder continuous potential which we shall denote ϕ^u as well, by an abuse of notation.

ϕ^u plays a central role in the applications of the thermodynamical formalism to Axiom A systems and it is related to the computation of several important dynamical indicators as Hausdorff dimension and rates of escape from a small neighborhood of a compact isolated f-invariant hyperbolic set. See [7], [29] and below.

We use (4) to prove the following. Let μ be a non atomic ergodic hyperbolic measure, then there exists a sequence of ergodic Borel probabilities μ_n supported on hyperbolic basic sets Ω_n such that

$$\mu_n \to \mu \quad \text{and} \quad \frac{h(\mu_n)}{\chi^+(\mu_n)} \to \frac{h(\mu)}{\chi^+(\mu)}, \tag{6}$$

where

$$\chi^+(\mu) = \int \sum_{\chi_i > 0} \chi_i \dim(E_i) d\mu, \tag{7}$$

is the average of positive Lyapunov exponents counted with their multiplicity.

This is interesting from the point of view of dimension theory since these quantities are dimension-like properties of the dynamics. See subsection 6.3.

Our methods also provide approximation results for *Lyapunov exponents*. Namely, let μ be a non atomic ergodic Borel probability preserved by a surface diffeomorphism, then there exists a sequence μ_n of ergodic measures supported on hyperbolic Cantor sets Ω_n such that

$$\chi^+(\mu_n) \to \chi^+(\mu). \tag{8}$$

In higher dimensions we can ask if the Lyapunov exponents of an ergodic non atomic hyperbolic Borel probability μ can be *approximated simultaneously* by hyperbolic Cantor sets, that is, if there exist *hyperbolic basic sets* Ω_n and ergodic measures μ_n such that $\mu_n \to \mu$ and $\chi_i(\mu_n) \to \chi_i(\mu)$ for every $i = 1, \cdots, k(\mu)$. This was done in [18] and [36] using *hyperbolic periodic orbits* as approximating sets.

Finally we show that the variational pressure $P^*(\phi^u)$ of a compact nonuniformly hyperbolic f-invariant isolated Λ, is the *supremum of the rates of escapes of uniformly hyperbolic basic sets* $\Omega \subset \Lambda$, only assuming that Λ supports a non atomic f-invariant hyperbolic measure. This is related to an important conjecture due to J.P. Eckmann and D. Ruelle. See subsection 6.4.

1.3. **Strategy of proof.** We prove Main Theorem by a limiting argument. We start proving that for every ergodic non atomic hyperbolic measure μ and every admissible potential $\Phi \in \mathcal{S}^+$ there exist a sequence of hyperbolic basic sets Ω_n with the strong approximation property (see below) such that

$$P^*(f|\Omega_n, \Phi) \to h(\mu) + \int \Phi d\mu. \tag{9}$$

This will be our Theorem 1 (see section 4). We prove Main Theorem by a 'diagonal' argument using Theorem 1. See section 4. Theorem 1 will be proved similarly by a limiting argument using

Main Technical Lemma *Let ϕ be continuous. Then, for every non atomic hyperbolic Borel probability μ there exists a sequence of basic sets Ω_n such that*

$$P(f|\Omega_n, \phi) \to h(\mu) + \int \phi d\mu \tag{10}$$

Furthermore, Ω_n has the following **strong approximation property***:*

$$\mu_n \to \mu \textit{ for every sequence of ergodic measures } \mu_n \in \mathcal{M}_f(\Omega_n).$$

Remark 1.2. As we shall see in subsection 7.2, strong approximation property says that, given an open neighborhood $\mathcal{U}$ of $\mu \in \mathcal{M}_f$ in the weak topology, there exists $N = N(\mathcal{U}) > 0$ such that

$$\mathcal{M}_f(\Omega_n) \subset \mathcal{U} \quad \text{for every} \quad n \geq N.$$

See Proposition 7.5. This was proved in a joint work with Stefano Luzzatto. See [19]. The strong approximation property generalizes several well known results providing approximations of a non atomic hyperbolic measure μ by ergodic measures satisfying some variational property, supported on suitable sequences of hyperbolic horseshoes and provide us with a useful tool to approximate dynamical indicators of nonuniformly hyperbolic systems, as we shall see below.

Remark 1.3. If $\Phi = \phi$ is continuous then Main Technical Lemma proves Theorem 1. Otherwise, there exists a sub(super)additive sequence $\{\phi_n\}$ such that

$$\int \frac{\phi_n}{n} d\mu \to \int \Phi d\mu \quad \forall\ \mu \in \mathcal{M}_f.$$

By Main Technical Lemma there exists, for every $n > 0$, a sequence $\{\Omega_m^n\}$ with the strong approximation property such that

$$P\left(f|\Omega_m^n, \frac{\phi_n}{n}\right) \to h(\mu) + \int \frac{\phi_n}{n} d\mu \quad \text{as} \quad m \to +\infty.$$

Here we develop a *'diagonal argument'* on the sequence Ω_m^n using in a fundamental way the strong approximation property and the semicontinuity of $\mu \longmapsto \int \Phi d\mu$, which follows from the sub(super)additive property of the sequence ϕ_n. See section 5.

Remark 1.4. Results similar to Main Technical Lemma has been mentioned previously in the litterature. In fact, as follows from [16, Corollary S.5.9] that, a non atomic hyperbolic measure μ there exists a sequence Ω_n of horseshoes and ergodic measures μ_n such that

$$h(\mu_n) + \int \phi d\mu_n \to h(\mu) + \int \phi d\mu \quad \text{for every continuous } \phi.$$

However one need to reshape [16, Theorem S.5.10] to arrive to Main Technical Lemma's conclusions, in particular the strong approximation property. To the best of my knowledge, no constructive proof has been given previously in the litterature for the existence of hyperbolic Cantor sets Ω_n satisfying our claims.

Main Technical Lemma proof will ocuppy the major part of the paper. For this we use that the free energy $E(\phi, \mu) = h(\mu) + \int \phi d\mu$ is a weighted rate of growing of dynamically non equivalent finite orbits, up to finite precision (see Proposition 7.7 in subsection 7.3). We borrow this idea from Mendoza's [22]. We then construct a horseshoe with finitely many branches and variable return time Ω^* (subsection 7.1) choosing suitable returns to a non invariant uniformly hyperbolic Pesin set which forms a representative sample of this statistic. Hyperbolic branches of Ω^* are quasi-generic in that finite orbits shadowed up to return time by these branches approximate μ with small finite precision (subsection 7.2). Let Ω be the f-invariant saturate of Ω^*. This is hyperbolic basic set (see subsection 7.1). We prove that $P(f|\Omega, \phi)$ is good approximation of free energy $E(\phi, \mu)$. For this we estimate the topological pressure of Ω as

$$P(\phi) = \lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{x\in Per(n)} \exp \sum_{j=0}^{n-1} \phi(f^j(x)) \right).$$

See section 11. Certain care has to be taken to keep track of the combinatorics of periodic orbits, due to the variable return times defining Ω^*. This is done in section 10.

1.4. **Organization of the paper.** The paper is organized as follows: in section 2 nonuniformly hyperbolic conditions are introduced with the definition of a regular nonuniformly hyperbolic diffeomorphism; section 3 defines the class of admissible potentials showing examples; main results with its applications are proved in sections 4, 5 and 6. The rest of the paper is dedicated to prove the main technical result. Sections 7 and 8 contains preliminary material. We give the arguments to choose Ω at section 9. The estimation of the topological pressure is done at sections 10 and 11.

Contents

2. Nonuniform hyperbolicity

Let f be a C^r $r \geq 1$ diffeomorphism of a compact manifold M. We define the *Lyapunov exponent* $\chi(x, v)$ by the limit

$$\chi^{\pm}(x, v) := \limsup_{n \to \pm\infty} \frac{\log \|df^n(x)v\|}{n}.$$

A point x is *Oseledec regular* if the limit exists and $\chi^+(x, v) + \chi^-(x, v) = 0$ for every $v \in T_x M - \{0\}$. Oseledec Theorem (see [20]) says that the set of regular points Λ is a Borel subset of total probability i.e. it has $\mu(\Lambda) = 1$, for every f-invariant Borel probability μ. We say that an invariant Borel probability μ is *hyperbolic* if it has non-zero Lyapunov exponents.

Definition 2.1. From now on we shall say that a diffeomorphism f of a compact Riemannian manifold M is *nonuniformly hyperbolic* if every ergodic f-invariant Borel probability μ is hyperbolic.

Given an f-invariant Borel probability measure μ, there exists a set Σ with $\mu(\Sigma) = 1$ for which the Lyapunov exponents $\chi(x, v)$ are well defined for every $x \in \Sigma$ and every $v \in T_x M \setminus \{0\}$ and the measure μ is hyperbolic if all the Lyapunov exponents are non-zero. Moreover, if μ is ergodic, as in our setting, then $\chi(x, v)$ can only take on a finite number of values on Σ. In that case, there exists a constant χ satisfying

$$\min\{|\chi(x, v)| : x \in \Sigma, v \in T_x M \setminus \{0\}\} > \chi > 0. \tag{11}$$

Then, for all sufficiently small $\epsilon > 0$ such that $\chi - \epsilon > 0$ (or equivalently $-\chi + \epsilon < 0$), by Oseledec's Theorem there exist measurable Df-invariant decompositions

$$T_x M = E^s(x) \oplus E^u(x),$$

and tempered Borel measurable functions $C_\epsilon, K_\epsilon : \Sigma \to (0, +\infty)$ such that

$$\begin{cases} \|Df^n(x)v\| \le C_\epsilon(x)e^{n(-\chi+\epsilon)}\|v\| & \forall\ v \in E^s(x)\ \forall\ n \ge 0 \\ \|Df^{-n}(x)v\| \le C_\epsilon(x)e^{n(-\chi+\epsilon)}\|v\| & \forall\ v \in E^u(x)\ \forall\ n \ge 0 \end{cases}$$

and $\angle(E^s(x), E^u(x)) \ge K_\epsilon(x)$, where

$$E^s(x) := \bigoplus_{\chi_i(x)<0} E_i(x) \quad \text{and} \quad E^u(x) := \bigoplus_{\chi_i(x)>0} E_i(x).$$

Moreover, by Tempering-Kernel Lemma [16, Lemma S.2.12], we may suppose that

$$(1+\epsilon)^{-1} \le \frac{C_\epsilon(f(x))}{C_\epsilon(x)}, \frac{K_\epsilon(f(x))}{K_\epsilon(x)} \le 1+\epsilon, \quad \mu - a.e.$$

We remark that the properties given above as a consequence of the hyperbolicity of μ can also be formulated without any reference to the measure μ and are essentially nonuniform versions of standard uniformly hyperbolic conditions, see [4, Theorem 6.6].

We refer the reader to [4] and [16] for an exposition of the ergodic theory of smooth dynamical systems with hyperbolic behavior. We also refer to section 8 for a brief summary of the definitions and properties of nonuniformly hyperbolic systems which shall be used in the proofs.

Existence of *non atomic hyperbolic measures* is crucial to prove (3) since, as Bowen's figure eye-like example shows, there are nonuniformly hyperbolic systems where every f-invariant Borel probability is concentrated at finitely many points. See [14, pp. 140] This motivates the following

Definition 2.2. We say that a nonuniformly hyperbolic $C^{1+\alpha}$ diffeomorphism f leaving invariant an ergodic *non atomic* hyperbolic measure μ is *regular nonuniformly hyperbolic.*

3. Admissible potentials

We recall that a sequence $\mathcal{F} = \{\phi_n\}$ is *subadditive (resp. superadditive)* if $\phi_{n+m} \le \phi_n + \phi_m \circ f^n$ (resp. $\phi_{n+m} \ge \phi_n + \phi_m \circ f^n$).

Definition 3.1. A Borel measurable real function Φ is an *admissible potential* if either it is a continuous or there exists a sub(super) additive sequence of continuous functions $\{\phi_n\}$ such that Φ is its rate of growing, that is

$$\Phi = \lim_{n\to+\infty} \frac{\phi_n}{n}, \tag{12}$$

pointwise limit in a set of total probability, that is, for μ almost everywhere for every f-invariant Borel probability $\mu \in \mathcal{M}_f$. We require in addition that $\|\Phi\|_{\infty,\mu} < +\infty$ for every f-invariant Borel probability $\mu \in \mathcal{M}_f$, where $\|\Phi\|_{\infty,\mu}$ is the μ-essential supremum.We denote $\mathcal{S}^+$ the set of admissible potentials.

Remark 3.1. Notice that $\mathcal{S}^+$ is a dense subset of $L^p(\mu)$, $p \ge 1$, for every f-invariant Borel probability $\mu \in \mathcal{M}_f$. Indeed, as $C(M)$ is dense in $L^p(\mu)$ then

$$\mathcal{S} = \left\{ \frac{\phi_n}{n} : n \in \mathbb{N},\, \phi_n \in C(M) \right\} \cup C(M) \subset L^p(\mu)$$

will be dense too. Therefore, the set of limit points, which is non empty by virtue of Kingman's Theorem, will be a nonempty dense subset of L^p, since convergence in almost every point implies convergence in L^p-norm.

The following version of main results in [17] was proved in [27][Theorem 1.1]:

Kingman's Subadditive Ergodic Theorem *Let $f : M \to M$ be a measurable map leaving invariant a Borel probability μ and $\mathcal{F} = \{\phi_n\}$ be a sub(super)additive sequence of measurable functions taking values on $\mathbb{R} \cup \{-\infty\}$ such that*

$$\phi_1^+ = \max\{\phi_1, 0\} \in L^1(\mu).$$

Then, there exists an f-invariant Borel measurable function $\phi : M \to real \cup \{-\infty\}$ such that $\phi^+ \in L^1(\mu)$ and

$$\frac{\phi_n}{n} \to \phi \quad \mu - a.e.$$

In the subadditive case,

$$\lim_{n\to+\infty} \frac{1}{n} \int \phi_n d\mu = \inf_{n>0} \frac{1}{n} \int \phi_n = \int \phi d\mu;$$

replacing 'inf' *for* 'sup' *for superadditive sequences.*

As mentioned in [8] there is number of interesting problems in dimension theory leading naturally to the study of thermodynamics of subadditive sequences arising as rates of growing of matrix valued cocycles. See [8, Corollary 1.2] and remarks therein.

The main examples of admissible potentials we have in mind are rates of growing of linear cocycles over f. Namely, let $\pi : E \to M$ be a smooth finite dimensional vector bundle over a compact (connected) Riemannian manifold M endowed with a smooth Finsler form $\|\|_E = \{\|\|_x : E_x \to \mathbb{R} : x \in M\}$. We suppose in addition that $\dim(E_x)$ is constant. Let $f : M \to M$ a C^r smooth diffeomorphism $r \geq 1$. We say that $F : E \to E$ is a *measurable cocycle of linear isomorphisms* covering f if:

(1) $\pi \circ F = f \circ \pi$;
(2) the map $x \mapsto F_x$ is a measurable section of the associated vector bundle $L(E, E)$ defined over a Borel subset;
(3) $F(x) := F|E_x : E_x \to E_{f(x)}$ is a non-singular linear operator.

If F^n denotes the n-fold composition of F with itself then

$$F^n(x) := F(f^{n-1}(x)) \circ \cdots \circ F(x)$$

for every $x \in M$ and for every $n \in \mathbb{Z}$.

Let $F : E \to E$ a measurable linear cocycle covering $f \in Diff^r(M)$, $r \geq 1$, over a Borel measurable subset of total probability and suppose

$$\mu - ess \sup_{x\in M} \log \|F(x)\|_x < +\infty \quad \text{for every} \quad \mu \in \mathcal{M}_f.$$

Then, the rate of growth

$$\Phi(x) := \lim_{n\to+\infty} \frac{\log \|F^n(x)\|_x}{n}$$

is an admissible potential $\Phi \in \mathcal{S}^+$.

This is an straighforward consequence of Kingman's Theorem since $\phi_n = \log \|F^n\|$ is subadditive and

$$\sup_{x\in M} \frac{\log \|F^n(x)\|_x}{n} \leq \sup_{x\in M} \log \|F(x)\|_x < +\infty \quad \mu - a.e. \quad \forall\, \mu \in \mathcal{M}_f.$$

Remark 3.2. Also, by the same arguments, the rates of growth

$$\Phi(x) := \lim_{n\to+\infty} -\frac{\log \|F^n(x)\|_x}{n} \in \mathcal{S}^+$$

define admissible potentials, since $\phi_n = -\log \|F^n\|$ is superadditive.

An important source of examples comes from the study of rates of growing of k-volumes under the derivative of a smooth map on a manifolds.

Let $E = \bigwedge^k TM$ be the vector bundle of k-volume forms over M, and $F(x) := \bigwedge^k Df(x)$ fiber map induced by the derivative $Df : TM \to TM$. Then, for every $1 \le k \le \dim(M)$ we define

$$\Phi^k := \lim_{n\to+\infty} \frac{1}{n} \log \| \bigwedge^k Df^n(x) \|. \tag{13}$$

The Borel measurable function Φ^k so defined is an admissible potential and it is the sum of the k-largest positive Lyapunov exponents, counted with their multiplicity. Similarly so, the rates of growing of k-volumes under the action of f^{-1} gives us the sum of the k-largest negative Lyapunov exponents. Compare [20] and [27].

In particular, for $k = \dim M$ we conclude that

$$\Phi^u = \lim_{n\to+\infty} -\frac{1}{n} \log \| \bigwedge^{\dim M} Df^n \| = -\sum_{\chi_i>0} \chi_i \dim E_i \tag{14}$$

is an admissible potential since $\{-\log \| \bigwedge^{\dim M} Df^n \|\}$ is superadditive.

4. Proof of Main Theorem

Let x be an Oseledec regular point. We recall that x is hyperbolic if all the Lyapunov exponents at x are non zero. The *rate of hyperbolicity of a hyperbolic regular point* x is defined as $\chi(x) := \min\{|\chi_i(x)|\}$, where $-\infty < \chi_1(x) < \cdots < \chi_k(x) < +\infty$ is the spectrum of Lyapunov exponents of x. See [4]. We define the *rate of hyperbolicity of an f-invariant set* Ω as

$$\chi(\Omega) := \inf_{x\in\Omega} |\chi(x)|$$

and the *rate of hyperbolicity of a measure* μ as the infimum of $\chi(\Lambda)$ taken over the family of compact f-invariant subsets Λ with $\mu(\Lambda) > 0$.

Our main result follows straigthforwardly from

Theorem 1. *Let f be a regular nonuniformly hyperbolic diffeomorphism of a compact Riemannian manifold, μ be a hyperbolic ergodic Borel probability with positive metrical entropy and $\Phi \in \mathcal{S}^+$ an admissible potential. Then there exists a sequence of hyperbolic basic sets Ω_n and a constant $\chi > 0$ such that:*

1. *the rate of hyperbolicity of Ω_n is bounded from below by $\chi > 0$;*
2. $\mu_n \to \mu$ **for every sequence of ergodic measures** *with supp$(\mu_n) \subseteq \Omega_n$;*
3. $P^*(f|\Omega_n, \Phi) \to h(\mu) + \int \Phi d\mu$.

Remark 4.1. (1) The lower bound $\chi > 0$ on the rate of hyperbolicity of the approximating basic sets Ω_n only depends on $\chi(\mu)$, the rate of hyperbolicity of μ.

(2) Theorem 1 generalizes [16, Theorem S.5.9]. Moreover, we get a sequence Ω_n of hyperbolic basic sets Ω_n which strongly approximate μ in the sense that every sequence of ergodic Borel probabilities $\mu_n \in \Omega_n$ converges weaklky to μ. See [19] and references therein.

Proof. **[Proof of Main Theorem]** We first notice that $P^*(\Phi) \ge P^*(f|\Omega, \Phi)$ for every compact f-invariant subset Ω. Therefore $P^*(\Phi) \ge \sup_{\Omega\in\mathcal{H}} P^*(f|\Omega, \Phi)$.

By hypotheses there exists a hyperbolic Borel probability μ with positive metrical entropy $h(\mu) > 0$, therefore

$$P^*(\Phi) = \sup_{h(\mu)>0} \left\{ h(\mu) + \int \Phi d\mu : \mu \in \mathcal{M}_f \right\}$$

Therefore we can find a sequence of hyperbolic non atomic ergodic measures μ_n such that

$$h(\mu_n) + \int \Phi d\mu_n \to P^*(\Phi). \tag{15}$$

By Theorem 1, for every $n > 0$ there exists a sequence of hyperbolic basic sets $\{\Omega_{n,m}\}_{m>0}$ which approximates μ_n uniformly as $m \to +\infty$ such that

$$P^*(\,\Omega_{n,m}\,,\,\Phi) \to h(\mu_n) + \int \Phi d\mu_n \quad \text{as} \quad m \to +\infty.$$

Choose now a 'diagonal' sequence Ω_n such that $P^*(\Omega_n, \Phi) \to P^*(\Phi)$. This proves the variational equation (3) completing the proof. □

Remark 4.2. Nonuniform hyperbolicity hypotheses can be relaxed with a different set of conditions. This is important because one would like to have a similar result under weaker hypotheses in order to prevent the exclusion of elliptic or parabolic behaviors. For instance one might require for Φ to be *regularly attainable* is there exists a sequence of hyperbolic measures μ_n with $h(\mu_n) > 0$ satisfying (15), without impose the condition for every f-invariant Borel probability to be hyperbolic as we do in this paper. Compare [11, Section 3] and [12].

5. Proof of Theorem 1

We will prove Theorem 1 using Main Technical Lemma and the following couple of simple lemmas

Lemma 5.1. *Let $\{\phi_n\}$ be a subadditive (resp. superadditive) sequence of continuous functions and*

$$\Phi = \inf_{n>0} \frac{\phi_n}{n} \quad \textit{(resp.} \quad \Phi = \sup_{n>0} \frac{\phi_n}{n}\textit{)}$$

its rate of growing. Then,

$$\nu \longmapsto \int \left(\frac{\phi_n}{n} - \Phi \right) d\nu \quad \textit{is lower (resp. upper) semicontinuous.} \tag{16}$$

Proof. Let $\{\phi_n\}$ be a subadditive sequence of continuous functions and Φ its rate of growing. Then,

$$\nu \longmapsto \int \left(\frac{\phi_n}{n} - \Phi \right) d\nu = \sup_{m>0} \int \left(\frac{\phi_n}{n} - \frac{\phi_m}{m} \right) d\nu$$

is the sup of a sequence of continuous linear functionals defined of the space of Borel probability measures, changing to inf for a superadditive sequence. □

Lemma 5.2. *Let $\{\phi_n\}$ be a sub(super)additive sequence of continuous functions, $\Phi \in \mathcal{S}^+$ its rate of growing and let Ω be a compact f-invariant subset. Then,*

$$P^*(f|\Omega, \Phi) = \lim_{n\to+\infty} P\left(f|\Omega, \frac{\phi_n}{n} \right). \tag{17}$$

Moreover, if $\{\phi_n\}$ is subadditive then,

$$P^*(f|\Omega, \Phi) = \inf_{n>0} P\left(f|\Omega, \frac{\phi_n}{n} \right) \quad \textit{(resp. 'sup' for superadditive sequences)}$$

We first shall prove Theorem 1 using Lemma 5.1 and Lemma 5.2 and then we prove the lemmas. Proof of Main Technical Lemma will occupy the rest of the paper.

Proof. **[Proof of Theorem 1]**
Let μ be an ergodic non atomic hyperbolic Borel probability and $\Phi \in \mathcal{S}^+$ an admissible potential the rate of growing of a sub(super)additive sequence $\{\phi_n\}$ of continuous functions. Fix $\epsilon > 0$ small.

We first choose and fix $n > 0$ large enough such that

$$\left| \int \frac{\phi_n}{n} d\mu - \int \Phi d\mu \right| < \epsilon. \tag{18}$$

and then, using *Main Technical Lemma*, a sequence $\{\Omega_m^n\}$ of hyperbolic basic sets with *strong approximation property* such that

$$\sup_{\nu \in \mathcal{M}_f(\Omega_m^n)} \left\{ h(\nu) + \int \frac{\phi_n}{n} d\nu \right\} \to h(\mu) + \int \frac{\phi_n}{n} d\mu \quad \text{as} \quad m \to +\infty. \tag{19}$$

Let $M > 0$ such that

$$\left| \sup_{\nu \in \mathcal{M}_f(\Omega_m^n)} \left\{ h(\nu) + \int \frac{\phi_n}{n} d\nu \right\} - h(\mu) - \int \frac{\phi_n}{n} d\mu \right| < \epsilon \quad \forall\ m \geq M. \tag{20}$$

Then,

$$h(\mu) + \int \Phi d\mu - 2\epsilon < P\left(f|\Omega_m^n, \frac{\phi_n}{n} \right) < h(\mu) + \int \Phi d\mu + 2\epsilon \quad \forall\ m \geq M. \tag{21}$$

Then adding and substracting Φ inside the variational equation for topological pressure $P(\phi_n/n)$ we get

$$h(\mu) + \int \Phi d\mu - 2\epsilon < P^*\left(f|\Omega, \Phi\right) + \sup_{\nu \in \mathcal{M}_f(\Omega_m^n)} \int \left(\frac{\phi_n(x)}{n} - \Phi \right) d\nu(x). \tag{22}$$

By Lemma 5.1 and (18), for the subadditive case, there exists an open neighborhood $\mathcal{U} = \mathcal{U}(\mu)$ of μ, in the weak topology, such that

$$\int \left(\frac{\phi_n}{n} - \Phi \right) d\mu - \epsilon < \int \left(\frac{\phi_n}{n} - \Phi \right) d\nu \leq \int \left(\frac{\phi_n}{n} - \Phi \right) d\mu \tag{23}$$

for every $\nu \in \mathcal{U}$.

Now, by the strong approximation property of the sequence Ω_m^n we can choose $m(\epsilon) \geq M$ such that

$$\mathcal{M}_f(\Omega_{m(\epsilon)}^n) \subset \mathcal{U} \tag{24}$$

Therefore, using (18), (23) and (24)

$$\sup_{\nu \in \mathcal{M}_f(\Omega_{m(\epsilon)}^n)} \int \left(\frac{\phi_n(x)}{n} - \Phi \right) d\nu(x) < \epsilon$$

and we thus have from (22) that

$$h(\mu) + \int \Phi d\mu - 3\epsilon < P^*\left(f|\Omega_{m(\epsilon)}^n, \Phi \right). \tag{25}$$

On the other hand, by (21) and Lemma 5.2

$$P^*(f|\Omega_{m(\epsilon)}^n, \Phi) = \inf_{k>0} P\left(f|\Omega_{m(\epsilon)}^n, \frac{\phi_k}{k} \right) \leq P\left(f|\Omega_{m(\epsilon)}^n, \frac{\phi_n}{n} \right) < h(\mu) + \int \Phi d\mu + 2\epsilon \tag{26}$$

thus proving that

$$\left| P^*(f|\Omega_{m(\epsilon)}^n, \Phi) - h(\mu) - \int \Phi d\mu \right| < 4\epsilon. \tag{27}$$

The superadditive case is treated similarly: by Lemma 5.1 there exists an open neighborhood $\mathcal{U} = \mathcal{U}(\mu)$ of μ, in the weak topology, such that

$$\int \left(\frac{\phi_n}{n} - \Phi\right) d\mu \leq \int \left(\frac{\phi_n}{n} - \Phi\right) d\nu < \int \left(\frac{\phi_n}{n} - \Phi\right) d\mu + \epsilon \tag{28}$$

for every $\nu \in \mathcal{U}$ and thus, choosing $m(\epsilon) \geq M$ sufficiently large we get the inclusion (24) and thus from (18) and (28)

$$\sup_{\nu \in \mathcal{M}_f(\Omega^n_{m(\epsilon)})} \int \left(\frac{\phi_n(x)}{n} - \Phi\right) d\nu(x) < 2\epsilon$$

Therefore, by (22), we have

$$h(\mu) + \int \Phi d\mu - 4\epsilon < P^* \left(f|\Omega^n_{m(\epsilon)}, \Phi\right). \tag{29}$$

Once again, by adding and substracting again Φ inside the variational formula for the topological pressure we get from (21)

$$P^* \left(f|\Omega^n_m, \Phi\right) + \inf_{\nu \in \mathcal{M}_f(\Omega^n_m)} \int \left(\frac{\phi_n(x)}{n} - \Phi\right) d\nu(x) \leq P\left(f|\Omega^n_{m(\epsilon)}, \frac{\phi_n}{n}\right) < h(\mu) + \int \Phi d\mu + 2\epsilon \tag{30}$$

and then, by (18), (24) and (28),

$$-\epsilon < \inf_{\nu \in \mathcal{M}_f(\Omega^n_{m(\epsilon)})} \int \left(\frac{\phi_n(x)}{n} - \Phi\right) d\nu(x),$$

so getting from (30)

$$P^* \left(f|\Omega^n_m, \Phi\right) < h(\mu) + \int \Phi d\mu + 3\epsilon. \tag{31}$$

Hence, from (29) and (31) we get again estimative (27).

As $\epsilon > 0$ is arbitrary we can take $\epsilon_n \downarrow 0^+$ so defining a sequence $\Omega_n := \Omega^n_{m_n}$ where $m_n = m(\epsilon(n))$ of hyperbolic basic sets with the strong approximation property such that:

$$P^*(f|\Omega_n, \Phi^*) \to h(\mu) + \int \Phi d\mu,$$

as we claimed. □

Proof. **[Proof of Lemma 5.2]**
Let $\{\phi_n\}$ be a subadditive continuous potential. Using subadditivity and lett ing $m = np + q$ for some $0 \leq q < n$, we have

$$\phi_m \leq \phi_{np} \circ f^q + \phi_q \leq \sum_{k=0}^{p-1} \phi_n \circ f^{q+kn} + qL$$

since $\|\phi_n\| \leq nL$ by assumption. Therefore, for every f-invariant Borel probability $\mu \in \mathcal{M}_f$

$$\begin{aligned} \int \frac{\phi_m}{m} d\mu &\leq \frac{1}{m} \int \sum_{k=0}^{p-1} \phi_n \circ f^{q+kn} d\mu + \frac{qL}{m} \\ &\leq \frac{np}{m} \int \frac{\phi_n}{n} d\mu + \frac{nL}{m}. \end{aligned}$$

Then, substituting into (1), the variational equation for topological pressure, we get

$$\begin{aligned} P\left(f|\Omega, \frac{\phi_m}{m}\right) &\leq P\left(f|\Omega, \frac{np}{m}\frac{\phi_n}{n} + \frac{nL}{m}\right) \\ &= P\left(f|\Omega, \frac{np}{m}\frac{\phi_n}{n}\right) + \frac{nL}{m}, \end{aligned}$$

using [32, Theorem 2.1, vii]. By [32, Theorem 2.1, v]

$$\left|P\left(f|\Omega, \frac{np}{m}\frac{\phi_n}{n}\right) - P\left(f|\Omega, \frac{\phi_n}{n}\right)\right| \leq \left(\frac{np}{m} - 1\right)\left\|\frac{\phi_n}{n}\right\|_\infty.$$

Then for every small $\epsilon > 0$ there exists $M = M(\epsilon) >> n$ such that

$$P\left(f|\Omega, \frac{\phi_m}{m}\right) \leq P\left(f|\Omega, \frac{\phi_n}{n}\right) + \epsilon \quad \text{for every} \quad m \geq M.$$

Therefore,

$$\limsup_{m\to+\infty} P\left(f|\Omega, \frac{\phi_m}{m}\right) \leq P\left(f|\Omega, \frac{\phi_n}{n}\right), \quad \text{for every} \quad n > 0.$$

since $\epsilon > 0$ is arbitrary and we thus have

$$\limsup_{m\to+\infty} P\left(f|\Omega, \frac{\phi_m}{m}\right) \leq \inf_{n>0} P\left(f|\Omega, \frac{\phi_n}{n}\right) \leq \liminf_{m\to+\infty} P\left(f|\Omega, \frac{\phi_m}{m}\right)$$

concluding that the limit exists and

$$\lim_{n\to+\infty} P\left(f|\Omega, \frac{\phi_m}{m}\right) = \inf_{n>0} P\left(f|\Omega, \frac{\phi_n}{n}\right).$$

From here we easily deduce that

$$P^*(f|\Omega, \Phi) = \inf_{n>0} \sup_{\nu\in\mathcal{M}_f(\Omega)} \left\{h(\nu) + \int \frac{\phi_n}{n} d\nu\right\}.$$

Actually, as

$$\int \Phi d\mu = \inf_{n>0} \int \frac{\phi_n}{n} d\mu \quad \text{for every} \quad \mu \in \mathcal{M}_f,$$

then, for every $n > 0$,

$$P^*(f|\Omega, \Phi) \leq \sup_{\nu\in\mathcal{M}_f(\Omega)} \left\{h(\nu) + \int \frac{\phi_n}{n} d\nu\right\}.$$

Hence,

$$P^*(f|\Omega, \Phi) \leq \inf_{n>0} \sup_{\nu\in\mathcal{M}_f(\Omega)} \left\{h(\nu) + \int \frac{\phi_n}{n} d\nu\right\} = \inf_{n>0} P\left(f|\Omega, \frac{\phi_n}{n}\right).$$

On the other side, we claim that, for any given small $\epsilon > 0$ there exists $N = N(\epsilon) > 0$ such that

$$\int \frac{\phi_n}{n} d\mu < \int \Phi d\mu + \epsilon, \qquad \forall\ n \geq N \quad \text{and} \quad \forall\ \mu \in \mathcal{M}_f.$$

This follows from a compacity argument applied to $\mu \in \mathcal{M}_f$ and that

$$\mu \longmapsto \int \Phi d\mu = \inf_{n>0} \int \frac{\phi_n}{n} d\mu \quad \text{is uppersemicontinuous.}$$

Therefore,

$$\sup_{\nu\in\mathcal{M}_f(\Omega)} \left\{h(\nu) + \int \frac{\phi_n}{n} d\nu\right\} \leq P^*(f|\Omega, \Phi) + \epsilon \quad \text{for every} \quad n \geq N.$$

so that,

$$\inf_{n>0} P\left(f|\Omega, \frac{\phi_n}{n}\right) \leq P^*(f|\Omega, \Phi),$$

since $\epsilon > 0$ is arbitrary.

The superadditive case follows from similar arguments. We start remarking that

$$\phi_m \geq \phi_{np} \circ f^q + \phi_q \geq \sum_{k=0}^{p-1} \phi_n \circ f^{q+kn} - qL,$$

and then that

$$\int \frac{\phi_m}{m} d\mu \geq \int \frac{np}{m} \frac{\phi_n}{n} d\mu - \frac{nL}{m} \quad \forall\ \mu \in \mathcal{M}_f.$$

This proves

$$P\left(f|\Omega, \frac{\phi_m}{m}\right) \geq P\left(f|\Omega, \frac{np}{m}\frac{\phi_n}{n}\right) - \frac{nL}{m}$$

and then, for a suitable $M >> n$, that

$$P\left(f|\Omega, \frac{\phi_m}{m}\right) \geq P\left(f|\Omega, \frac{\phi_n}{n}\right) - \epsilon \quad \text{for every} \quad m \geq M.$$

Therefore,

$$\liminf_{m\to+\infty} P\left(f|\Omega, \frac{\phi_m}{m}\right) \geq \sup_{n>0} P\left(f|\Omega, \frac{\phi_n}{n}\right) \geq \limsup_{n\to+\infty} P\left(f|\Omega, \frac{\phi_n}{n}\right),$$

proving that that the limit exists and is

$$\lim_{n\to+\infty} P\left(f|\Omega, \frac{\phi_n}{n}\right) = \sup_{n>0} P\left(f|\Omega, \frac{\phi_n}{n}\right).$$

Hence,

$$\begin{aligned}
\sup_{n>0} P\left(f|\Omega, \frac{\phi_n}{n}\right) &= \sup_{n>0} \sup_{\nu\in\mathcal{M}_f(\Omega)} \left\{h(\nu) + \int \frac{\phi_n}{n} d\nu\right\} \\
&= \sup_{\nu\in\mathcal{M}_f(\Omega)} \left\{h(\nu) + \sup_{n>0} \int \frac{\phi_n}{n} d\nu\right\} \\
&= \sup_{\nu\in\mathcal{M}_f(\Omega)} \left\{h(\nu) + \int \Phi d\nu\right\} \\
&= P^*(f|\Omega, \Phi).
\end{aligned}$$

□

6. Applications

We would like to show some applications of the Main Theorem and Theorem 1 before we go into the tedious details of the proof of Main Technical Lemma.

6.1. **Pressures.** As an inmediate application of Main Theorem and Theorem 1 we would like to mention the following generalizations of [16, Theorem S.5.9] and its corollary [16, Corollary S.5.10] and [14, Corollary 4.4].

Theorem 2. *Let $f : M \to M$ be a nonuniformly hyperbolic $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold. Suppose in addition that f leaves invariant at least one non atomic hyperbolic Borel probability and let $\phi \in C(M)$ a continuous potential. Then,*

$$P(\phi) = \sup_{\Omega\in\mathcal{H}} P(f|\Omega, \phi). \tag{32}$$

In particular, for every continuous potential ϕ there exists a sequence of hyperbolic basic sets Ω_n such that $P(f|\Omega_n, \phi) \to P(\phi)$.

Corollary 6.1. *Under the same hypotheses of Theorem 2 there exists a sequence Ω_n of hyperbolic basic sets such that*

$$h(f) = \sup_{\Omega \in \mathcal{H}} h(f|\Omega). \tag{33}$$

Similarly so, if $h(f) > 0$ and f is nonuniformly hyperbolic then there exists a sequence of hyperbolic basic sets Ω_n such that $h(f|\Omega_n) \to h(f)$. Compare [14, Corollary 4.4].

Theorem 3. *Let $f : M \to M$ be a nonuniformly hyperbolic $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold. Suppose in addition that f leaves invariant at least one non atomic hyperbolic Borel probability, then:*

$$\sup_{\mu \in \mathcal{M}_f} \{h(\mu) - \chi^+(\mu)\} = \sup_{\Omega \in \mathcal{H}} P(f|\Omega, \phi^u) \tag{34}$$

Proof. First notice that ϕ^u is μ-summable for every f-invariant Borel probability μ and that

$$\lim_{n \to +\infty} \frac{1}{n} \phi^u(f^n(x)) = \Phi^u(x), \qquad \mu - a.e. \quad \forall\ \mu \in \mathcal{M}_f.$$

where $\Phi^u(x) = -\sum_{\chi_i(x)>0} \chi_i(x) \dim E_i(x)$, that is,

$$\int \Phi^u d\mu = -\chi^+(\mu).$$

Then, by Birkhoff's Theorem

$$\int \phi^u(x) d\mu(x) = \int \Phi^u(x) d\mu(x) \qquad \forall\ \mu \in \mathcal{M}_f. \tag{35}$$

Therefore,

$$\begin{aligned} \sup_{\mu \in \mathcal{M}_f} \{h(\mu) + \int \Phi^u d\mu\} &= \sup_{\Omega \in \mathcal{H}} P(f|\Omega, \Phi^u) \\ &= \sup_{\Omega \in \mathcal{H}} P(f|\Omega, \phi^u) \end{aligned}$$

using (35) and the variational principle (3). This proves (34) since

$$\sup_{\mu \in \mathcal{M}_f} \{h(\mu) - \chi^+(\mu)\} = \sup_{\mu \in \mathcal{M}_f} \{h(\mu) + \int \Phi^u d\mu\}$$

□

It follows from (35) that $P^*(\phi^u) = P^*(\Phi^u)$, thus giving sense to the variational pressure of the unstable potential ϕ^u, eventhough ϕ^u do not belong to $\mathcal{S}^+$, the class of admissible potentials, except for Axiom A systems.

Corollary 6.2. *Let f be a regular nonuniformly hyperbolic diffeomorphism. Then*

$$P^*(\phi^u) = \sup_{\Omega \in \mathcal{H}} P(f|\Omega, \phi^u), \tag{36}$$

where ϕ^u is the unstable potential.

6.2. **Lyapunov exponents.**

Theorem 4. *Let f be a C^∞ regular nonuniformly hyperbolic surface diffeomorphism and μ a hyperbolic non atomic hyperbolic Borel probability. Then, there exists a sequence of ergodic measures μ_n, supported on hyerbolic Cantor sets Ω_n, such that*

$$\chi^+(\mu_n) \to \chi(\mu). \tag{37}$$

Remark 6.1. We use the assumption on the smoothness of f to assure that $\mu \mapsto h(\mu)$ is upper semicontinuous. See [24].

Lemma 6.1. *The positive Lyapunov exponent $\chi^+ = \chi^+(x)$ is an admissible potential.*

Proof. Recall that

$$\chi^+(x) = \lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(x)\| \quad \mu - a.e.\ \forall\ \mu \in \mathcal{M}_f.$$

Then the conclusion follows since $\{\log \|Df^n(x)\|\}$ is subadditive. □

Lemma 6.2.

$$\mu \longmapsto \chi^+(\mu) := \int \chi^+(x)d\mu(x)$$

is upper semicontinuous.

Proof.

$$\chi^+(\mu) = \inf_{n>0} \int \frac{1}{n} \log \|Df^n\| d\mu$$

is the infimum of a sequence of continuous functionals defined over the compact metric space $\mathcal{M}_f$. □

Proof. **[Proof of Theorem 4]**
By Theorem 1 there exists a sequence of hyperbolic Cantor sets Ω_n such that $\mu_n \to \mu$ for every sequence μ_n of ergodic measures supported on Ω_n and

$$P^*(f|\Omega_n, \chi^+) = \sup_{\nu\in\mathcal{M}_f(\Omega_n)} \{h(\nu) + \int \chi^+ d\nu\} \to h(\mu) + \chi^+(\mu)$$

Hence, given $\epsilon > 0$ we find $N > 0$ such that for every $n \geq N$ there exists an ergodic Borel probability $\mu_n = \mu_n^\epsilon \in \Omega_n$ such that

$$h(\mu) + \chi^+(\mu) - \epsilon < h(\mu_n) + \chi^+(\mu_n).$$

By the *strong approximation property* we have that $\mu_n \to \mu$. Thus, by the semicontinuity of the entropy, we have –taking $N > 0$ larger if necessary– that $h(\mu_n) < h(\mu) + \epsilon$ for every $n \geq N$. Therefore,

$$\chi^+(\mu) - 2\epsilon < \chi^+(\mu_n) \quad \text{for every} \quad n \geq N.$$

Now, take $\epsilon_n \downarrow 0^+$ any sequence of positive numbers decreasing to zero and, relabeling $\mu_n = \mu_n^{\epsilon_n}$ we get a sequence of ergodic measures μ_n with $supp(\mu_n) \subset \Omega_n$ such that $\mu_n \to \mu$ –by the strong approximation property– and

$$\chi^+(\mu) \leq \liminf_{n\to+\infty} \chi^+(\mu_n) \leq \limsup_{n\to+\infty} \chi^+(\mu_n) \leq \chi^+(\mu),$$

thus proving

$$\chi^+(\mu_n) \to \chi^+(\mu).$$

□

Remark 6.2. In the higher dimensional setting, the above arguments prove that, for every $0 < k \le \dim(M)$ the sum of the k-largest positive Lyapunov exponents, counted with their multiplicity,

$$\chi_k^+(\mu) := \int \sum_{k-\text{largest}\, \chi_i > 0} \chi_i(x) \dim(E_i(x)) d\mu(x)$$

can be approximated by sequences of hyperbolic Cantor sets. Simply apply Theorem 1 to the admissible potential

$$\Phi^k := \lim_{n \to +\infty} \frac{1}{n} \log \| \bigwedge^k Df^n(x) \|.$$

6.3. **Dimensional characteristics.**

Theorem 5. *Let μ be an ergodic Borel probability preserved by a regular nonuniformly hyperbolic diffeomorphism. Then there exists a sequence of hyperbolic Cantor sets Ω_n with rate of hyperbolicity bounded from below by a constant $\chi > 0$ and ergodic measures μ_n supported on Ω_n such that $\mu_n \to \mu$ weakly and*

$$\frac{h(\mu_n)}{\chi^+(\mu_n)} \to \frac{h(\mu)}{\chi^+(\mu)} \tag{38}$$

Proof. Let $0 < d \le 1$ such that $h(\mu) - d\chi^+(\mu) = 0$. Then,

$$h(\mu) + \int \Phi_d^u d\mu = 0 \quad \text{where} \quad \Phi_d^u := \lim_{n \to +\infty} -\frac{1}{n} \log \| \bigwedge^{\dim(M)} Df^n(x) \|^d.$$

Then, by Theorem 1 there exists a sequence Ω_n of approximating hyperbolic sets for μ such that

- $\mu_n \to \mu$ weakly for every sequence $\mu_n \in \Omega_n$ and
- $P^*(f|\Omega_n, d\phi^u) \to 0$,

since $\mu(\Phi_d^u) = \mu(d\phi^u)$ for every $\mu \in \mathcal{M}_f$. By thermodynamics of Axiom A systems we find, for every Ω_n, an equilibrium state μ_n for $d\phi^u|\Omega_n$ such that $\mu_n \to \mu$ and

$$h(\mu_n) - d\chi^+(\mu_n) \to 0.$$

Let $\rho_n = |h(\mu_n) - d\chi^+(\mu_n)|$. Then, as $\chi^+(\mu_n) \ge \chi > 0$, we have that

$$\left| \frac{h(\mu_n)}{\chi^+(\mu_n)} - d \right| = \frac{\rho_n}{\chi^+(\mu_n)} \le \frac{\rho_n}{\chi}, \quad \forall\, n > 0$$

thus proving (6). $\square$

Definition 6.3. We call

$$\dim_u(\mu) := \frac{h(\mu)}{\chi^+(\mu)} \tag{39}$$

the *unstable dimension of μ*.

Let f be a conformal diffemorphism, that is, there exists a positive continuous function $a = a(x)$ such that $Df(x) = a(x) id_{T_xM}$. Then if μ is an ergodic non atomic hyperbolic measure preserved by f then $\dim_u(\mu)$ is the Hausdorff dimension of the quotient measure of μ when projected along a measurable partition subordinated to the lamination of Pesin's stable leaves. See [21] and [34].

Let Ω be a hyperbolic basic set Ω of a *conformal diffeomorphism.* Then d, the solution to Bowen equation

$$P(f|\Omega, d\phi^u) = 0,$$

is the Hausdorff dimension of its unstable Cantor sets, that is, $d = \dim_{\mathcal{H}}(\Omega \cap W^u_{loc}(x))$. Moreover, let μ the equilibrium state for the Hölder continuous potential $d\phi^u$, then

$$\dim_u(\mu) = \dim_{\mathcal{H}}(\Omega \cap W^u_{loc}(x)).$$

See [29].

Therefore, (6) provides an approximation for this dynamical indicator in the spirit of Katok-Mendoza's results. More precisely, we have the

Corollary 6.3. *Let f be a regular nonuniformly hyperbolic diffeomorphism. Suppose in addition that f is* conformal. *Let μ be an ergodic non atomic hyperbolic measure. Then, there exists a sequence Ω_n of hyperbolic Cantor sets and ergodic measures μ_n such that*

$$\mu_n \to \mu \quad \text{and} \quad \dim_u(\mu_n) \to \dim_u(\mu). \tag{40}$$

This generalizes [22, Theorem 4.1] which says that an ergodic hyperbolic Borel probability μ preserved by a surface diffeomorphism satisfying Pesin's entropy formula $h(\mu) = \chi^+(\mu)$ can be approximated by ergodic measures supported on horseshoes with arbitrarily large unstable Hausdorff dimension.

Remark 6.3. For *non conformal diffeomorphisms*, the unstable dimension of a non atomic hyperbolic measure is not a geometrical invariant but just a dimension-like property of the dynamics. Notwithstanding its lack of intrinsical geometrical meaning, this dynamical dimension can be used to give useful bounds for the Hausdorff dimension of non-conformal dynamically defined Cantor sets. We used this method in [30] to generalize [22, Theorem 4.1] for non conformal diffeomorphisms.

6.4. **Rates of escapes.** Let Λ be a compact f-invariant locally maximal subset, that is, there exists an isolating neighborhood U containing Λ, such that $\Lambda = \bigcap_{-\infty}^{+\infty} f^n(U)$. Then, the *rate of escape of Λ from U* is defined as

$$\rho(U) := \limsup_{m \to +\infty} \frac{1}{m} \log(\mathrm{Vol}\,(U_m)), \tag{41}$$

where

$$U_m = \{\, x \in U \ : \ f^k(x) \in U \text{ for } k = 0, \cdots, m-1 \text{ and } f^m(x) \notin U \,\}$$

is the set of points in U which escapes from U after m iterates.

It is well known from the thermodynamics of Axiom A systems that $P(f|\Omega, \phi^u) = \rho(U)$ is the rate of escape from a suitable small neighborhood U of an isolated uniformly hyperbolic set Ω. See [7].

In [9] Eckmann and Ruelle raised the following important question concerning the relationship between topological pressure and rates of escapes from isolated subsets.

Eckmann-Ruelle Conjecture

Let $\Lambda \subset M$ be a locally maximal compact f-invariant set of a smooth diffeomorphism of a compact Riemannian manifold and suppose that there exists an ergodic measure μ which is an equilibrium state for the unstable potential ϕ^u, that is,

$$h(\mu) - \chi^+(\mu) = \sup_{\nu \in \mathcal{M}_f} \{h(\nu) - \chi^+(\nu)\}. \tag{42}$$

Then

$$h(\mu) - \chi^+(\mu) = \rho(U) \tag{43}$$

is the rate of escape of a sufficiently small isolating neighborhood U of Λ.

Remark 6.4. An ergodic measure μ satisfying (42) is called a *generalized Sinai-Ruelle-Bowen (SRB) measure*, according to Ruelle. See for instance [28].

It is not known how generally the conjecture holds beyond systems satisfying Axiom A. See [5] for a review of the problem.

Actually, Bonatti, Baladi and Schmitt constructed in [5] nonuniformly hyperbolic counterexamples to Eckmann-Ruelle's conjecture, that is: compact invariant nonuniformly hyperbolic locally maximal set Λ with an isolating neighborhood U such that

$$\rho(U) = 0 \quad \text{such that} \quad \sup_{\mu \in \mathcal{M}_f} \left\{ h(\mu) - \int \Phi^u d\mu \right\} < 0.$$

These sets are fabricated from a plug of an eye-like Bowen-Katok's examples mentioned before into an uniformly hyperbolic repellor and they all have the property that *every f invariant ergodic measures supported on Λ is concentrated on finitely many fixed points or periodic orbits.*

Our claim is that, except for that cases, Eckmann-Ruelle's conjecture should be true for *regular nonuniformly hyperbolic isolated subsets*, that is, compact f-invariant isolated nonuniformly hyperbolic subsets which can be approximated from its interior by increasing sequences of hyperbolic Cantor sets. This is precisely the case of f-invariant isolated nonuniformly hyperbolic subsets supporting non atomic hyperbolic measures.

Theorem 6. *Let $\Lambda \subset M$ be a nonuniformly hyperbolic locally maximal compact f-invariant subset of a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold. Suppose that Λ supports an* non atomic hyperbolic measure μ. *Then,*

$$\sup_{\mu \in \mathcal{M}_f} \{h(\mu) - \chi^+(\mu)\} = \sup_{\Omega \in \mathcal{H}} \rho(U_\Omega), \tag{44}$$

where $\rho(U_\Omega)$ is the rate of escape from a small isolating neighborhood U_Ω of $\Omega \in \mathcal{H}$.

Proof. By the ergodic theory of Axiom A systems (see [7])

$$\rho(U_\Omega) = P(f|\Omega, \phi^u)$$

Then (44) follows inmediatly from this remark and Theorem 3. □

It can proved using the methods of this paper that, under the conditions of Theorem 6, there exists an increasing sequence of transitive isolated hyperbolic Cantor sets $\Omega_n \subset \Lambda$ such that:

- $\Omega_n \subset \Lambda$;
- $\Omega_n \subset \Omega_{n+1}$ for every $n > 0$;
- $\Lambda = \overline{\bigcup_n \Omega_n}$ and
- every Ω_n has an compact isolating neighborhood U_n such that $U = \overline{\bigcup_n U_n}$ is a compact isolating neighborhood of Λ.

We call $\{\Omega_n\}$ a *regular exhaustion of* Λ.

Conjecture *Let $\Lambda \subset M$ be a locally maximal compact f-invariant subset of a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold. Suppose that Λ has a regular exhaustion. Then,*

$$\rho(U) = \sup_{n>0} \rho(U_n) \tag{45}$$

This would prove that Eckmann-Ruelle's conjecture holds true for *regular* nonuniformly hyperbolic isolated subsets.

7. Proof of Main Technical Lemma: preliminaries

7.1. **The geometrical model: Alekseev sets.** Our geometric model will be defined by a finite collection $\mathcal{S}$ of pairwise disjoint *stable cylinders* $\{S_1, .., S_N\}$ and corresponding pairwise disjoint

collection $\mathcal{U}$ of *unstable cylinders* $\{U_1,...,U_N\}$ contained in a rectangle R which are the domain (resp. co-domain) of suitable *hyperbolic branches*

$$f^{R_i} : S_i \to U_i$$

defined by finitely many return times R_i.

Definition 7.1. An *Alekseev set* Ω^* is defined by an array of hyperbolic branches $\{f^{R_i} : S_i \to U_i\}$ all whose stable cylinders S_i *'crosses'* all U_i's transversally and such that every U_i *'crosses'* all S_i's transversally. Ω^* is the maximal invariant set in R under iterations of f^R and its inverse

$$\Omega^* := \bigcap_{n\in\mathbb{Z}} (f^R)^n(\mathsf{R}),$$

where $f^R : \bigcup_i S_i \to \bigcup_i U_i$ is the piecewise smooth invertible map define by

$$f^R|_{S_i} := f^{R_i}|_{S_i} \quad \text{and} \quad (f^R)^{-1}|_{U_i} := f^{-R_i}|_{U_i}.$$

Remark 7.1. This construction was originated in the work of M. V. Alekseev aiming at describing topological analogues to Markov chains. See [1].

Lemma 7.2. *Ω^* is an f^R-invariant Cantor set endowed with a hyperbolic product structure defined by suitable laminations of local f^R-invariant manifolds $\mathcal{F}^S$ and $\mathcal{F}^U$.*

These hyperbolic Cantor sets are the primary blocks in our construction of approximating basic sets.

Proposition 7.3. *Let Ω^* an Alekseev set defined by finitely many hyperbolic branches f^{R_i} : $S_i \to U_i$, then:*

$$\Omega = \bigcup_i \bigcup_{j=0}^{R_i-1} f^j(\Omega_i^*), \tag{46}$$

is the f-invariant saturate of Ω^* and it is a topologically transitive, locally maximal, uniformly hyperbolic f-invariant subset.

7.2. **Strong approximation property.** Sequences Ω_n approximate uniformly μ in that $\mu_n \to \mu$ for every sequence μ_n of ergodic measures such that *supp*$\mu \subset \Omega_n$. Actually, given an open basic neighborhood $\mathcal{N}$ of μ in the weak-* topology our methods allows to construct hyperbolic basic sets $\Omega = \Omega(\mathcal{N})$ such that $\nu \in \mathcal{N}$ for every ergodic Borel probability ν supported on Ω. This is done as follows.

Let μ be a Borel probability satisfying our main hypotheses. First recall that a point x is generic for μ if

$$\frac{1}{n}\sum_{j=0}^{n-1} \phi(f^j(x)) \to \int \phi d\mu \quad \text{as} \quad n \to \infty \quad \text{for all continuous functions } \phi \in C^0(M).$$

Given a countable dense subset $\{\psi_i\}$ of $C^0(M)$ we denote, given two constants $\rho, s > 0$, the weak-$*$ (ρ, s) neighborhood of μ

$$\mathcal{O}(\mu, \rho, s) := \{\nu : \left|\int \psi_i d\mu - \int \psi_i d\nu\right| < \rho,\ i = 1, \cdots, s\}; \tag{47}$$

Clearly, $\mu_n \to \mu$ in the weak-* topology if and only if there are sequences $\rho_n \to 0^+$ and $s_n \to +\infty$ such that $\mu_n \in \mathcal{O}(\mu, \rho_n, s_n)$.

Definition 7.4. We say that a point x is (ρ, s, n) *quasi-generic* for the measure μ if

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(f^j(x)) - \int \phi_i d\mu \right| \leq \rho \quad \forall i \leq s.$$

Furthermore, we say that a hyperbolic branch

$$f^n : S \to U$$

is (ρ, s)-*quasi-generic* for μ if every $x \in S$ is (ρ, s, n) quasi-generic for μ.

Proposition 7.5. *Let $\rho, s > 0$ and suppose there exists an Alekseev set $\Omega^*(\rho, s)$ defined by (ρ, s) quasi-generic branches. Then $\mu_\Omega \in \mathcal{O}(3\rho, s)$ for every f-invariant ergodic probability measure μ_Ω supported on $\Omega(\rho, s)$, the f-invariant saturate of $\Omega^*(\rho, s)$. In particular,*

$$\mathcal{M}_f(\Omega(\rho, s)) \subset \mathcal{O}(3\rho, s),$$

where $\mathcal{M}_f(\Omega(\rho, s))$ denotes the set of f-invariant Borel probabilities supported on $\Omega(\rho, s)$.

We refer the reader to [19] for details.

7.3. **Thermodynamic formalism revisited.** Let us start recalling the following terminology:

(1) $E \subset X$ is an (ϵ, n)-*spanning set in* X if for every $x \in X$ there exists $y \in E$ such that $d(f^k(x), f^k(y)) \leq \epsilon$, for every $0 \leq k \leq n-1$;
(2) $E \subset X$ is an (ϵ, n)-*separated set in* X if for every pair of different points $x \neq y$ in E it holds $d(f^k(x), f^k(y)) > \epsilon$ for some $0 \leq k \leq n-1$;
(3) given an f-invariant Borel probability μ and a positive number $0 < \alpha < 1$, we say that E is an (ϵ, n, α)-spanning set for μ if

$$\mu\left(\bigcup_{x \in E} B(x, \epsilon, n)\right) \geq \alpha,$$

where

$$B(x, \epsilon, n) := \{\, y \in X \,:\, \mathrm{dist}(f^j(x), f^j(y)) < \epsilon,\ j = 0, \cdots, n-1 \,\}.$$

E is (ϵ, n)-spanning in X if and only if $M \subset \bigcup_{x \in E} B(x, \epsilon, n)$. Also notice that any maximal (ϵ, n)-separated set in X is (ϵ, n)-spanning.

Definition 7.6. Let $f : X \to X$ continuous and μ and f-invariant Borel probability an ϕ continuous. We define the *measure-theoretical pressure of ϕ w.r.t. μ* as

$$P_\mu(\phi) := \lim_{\alpha \to 0^+} \lim_{\epsilon \to 0^+} \lim_{n \to +\infty} \frac{1}{n} \log \left(\inf_E \left\{ \sum_{x \in E} \exp S_n \phi(x) \right\} \right), \tag{48}$$

where μ is an f-invariant Borel probability and the *infimum taken over (ϵ, n, α)-spanning subsets* $E \subset M$.

Proposition 7.7. *Let $f : X \to X$ a continuous self map of a compact metric space (X, d), ϕ continuous and $\mu \in \mathcal{M}_f$ an ergodic f-invariant Borel probability. Then,*

$$P_\mu(\phi) = h(\mu) + \int \phi d\mu. \tag{49}$$

See [22][Theorem 1.1].

Remark 7.2. Let $f : X \to X$ a continuous selfmap of a complete metric space (X, d) and let $\mathcal{F} = \{\phi_n\}$ be a sequence of continuous real functions. We define the *nonadditive topological pressure of* $\mathcal{F}$ as

$$P(\mathcal{F}) := \lim_{\epsilon\to 0^+} \lim_{n\to+\infty} \frac{1}{n} \log \left(\inf_E \left\{ \sum_{x\in E} \exp \phi_n(x) \right\} \right), \tag{50}$$

infimum taken over (ϵ, n)*-spanning subsets* $E \subset M$.

If we let ϕ be continuous and define $S_n\phi(x) := \sum_{j=0}^{n-1} \phi(f^j(x))$, then $P(\{S_n\phi\}) = P(\phi)$ is the additive topological pressure. The next proposition is [8, Theorem 1]:

Proposition *Let* $f : X \to X$ *a continuous self map of a compact metric space* (X, d) *and* $\mathcal{F} = \{\phi_n\}$ *a subadditive sequence of continuous functions. Suppose in addition that the rate of growing is uniformly bounded from below*

$$\inf_{n>0} \frac{\phi_n}{n} > -\infty.$$

Let Φ *be the rate of growing of* $\mathcal{F}$*. Then,*

$$P(\mathcal{F}) = \sup_{\mu\in\mathcal{M}_f} \left\{ h(\mu) + \int \Phi d\mu \right\}.$$

This shows that for Φ, the rate of growing of a subadditive sequence $\mathcal{F} = \{\phi_n\}$, $P^*(\Phi)$ is the subadditive topological pressure $P(\mathcal{F})$, a well-defined topological invariant of the dynamics.

In the superadditive case, as it was proved in Lemma 5.2, we have

$$P^*(\Phi) = \sup_{n>0} P\left(\frac{\phi_n}{n}\right).$$

However, the question remains of whether or not we can express the variational pressure (2) as a single variational principle englobing both the subadditive and superadditive potentials.

Remark 7.3. An earlier temptative to extend the notion of topological pressure was made by Falconer in [10], motivated by the application of thermodynamics to the study of fractal dimensions for non-conformal transformations. Falconer introduced a new notion of topological pressure $P(\mathcal{F})$ for subadditive sequences $\mathcal{F} = \{\phi_n\}$ of continuous functions (see next section for definitions). Then a variational principle like (1) were proved under some regularity of the sequence $\mathcal{F}$. Namely, let Λ be a topologically mixing repeller. It is assumed that there exists constants $M, a, b > 0$ such that

$$\frac{1}{n}|\phi_n(x)| \le M, \quad \frac{1}{n}|\phi_n(x) - \phi_n(y)| \le a|x - y|, \quad \forall\, x, y \in \Lambda,\, n \in \mathbb{N}$$

and $|\phi_n(x) - \phi_n(y)| \le b$ whenever x, y belong to the same n-cylinder of the geometrical construction of Λ originated in a Markov partition.

Later, Barreira extended in [2] previous results of Pesin and Pitskel [25] defining a topological pressure as a sort of dimensional characteristic using Carathéodory's method. Also in this case, a variational principle was established with an additional hypotheses on the sequence $\{\phi_n\}$. Concretely, $\phi_{n+1} - \phi_n \circ f$ should converge uniformly to a continuous function ψ in order to get a variational equation like (1). These ideas has been used succesfully in the analysis of dimensional characteristics of non conformal maps. See [3] for a recent account of applications from thermodynamics to the dimension theory of dynamical systems.

More recently Cao, Feng and Huang [8] provided a proof of a subadditive variational principle *without further hypotheses of regularity or uniformity of limits for* ϕ_n using (50). Their definition

is equivalent to Falconer's approach for a mixing repeller and to Barreira's definitions, under the assumption that $\mathcal{F} = \{\phi_n\}$ has tempered distortion. See [8, Proposition 4.7]

8. Pesin theory and Markovian coverings

As usual in nonuniformly hyperbolic dynamics we recall some technical facts from the smooth ergodic theory to prepare the setting for our proofs. We refer the reader to [16, 4] for details.

8.1. **Regular neighborhoods and rectangles.** The fundamental starting point for understanding and working with the geometric structure of systems with non-zero Lyapunov exponents is the notion of *regular neighborhood*. Our first step is to introduce a linear coordinate system $L(x) : \mathbb{R}^n \to T_xM$, for every $x \in \Sigma$, such that $A(x)$, the representative of the derivative $Df(x)$ in the new coordinates, is a diagonal block matrix adapted to Oseledec's decomposition. The map $x \in \Sigma \mapsto L(x) \in GL_n(\mathbb{R})$ is Borel measurable and the coordinate changes are *tempered*:

$$\limsup_{n\to\pm\infty} \frac{1}{n} \max\{\log \|L(f^n(x))\|, \log \|L^{-1}(f^n(x))\|\} = 0 \quad \mu\text{-a.e.}$$

Let $< \cdot\,, \cdot >$ denote the standard inner product in $\mathbb{R}^n$, then a new measurable *Lyapunov metric* $< \cdot\,, \cdot >'_x$ is defined so that $L(x) : (\mathbb{R}^n, < \cdot\,, \cdot >) \to (T_xM, < \cdot\,, \cdot >'_x)$ is a linear isometry and such that

$$e^{-\chi-\epsilon} \leq \|A^u(x)^{-1}\|, \|A^s(x)\| \leq e^{-\chi+\epsilon} \quad \mu\text{-a.e.},$$

where A^s and $A^u(x)$ are restrictions of $A(x)$ to the stable and unstable subspaces, respectively. The new norm is equivalent to the Riemannian metric g_x, bounded by a measurable tempered correction $D(x)$ depending only on the Riemannian structure of M and $\angle(E^s(x), E^u(x) >$:

$$\frac{\|v\|_x}{\sqrt{2}} \leq \|v\|'_x \leq D(x)\|v\|_x \quad \text{for every} \quad v \in T_xM - \{0\}.$$

This process is known as *ϵ-reduction*, see [16, Theorem S.2.10] and [4, §5.5].

Our second step is to introduce coordinate systems in which the dynamics is essentially uniformly hyperbolic. The domain in which these local coordinate systems apply are called regular neighborhoods. For this we define *Lyapunov charts*

$$\psi_x := \exp_x \circ L(x)$$

where $\exp_x : B(0, r_M) \subset T_xM \to M$ are local geodesic coordinates and $r_M > 0$ the injectivity radius of (M, g). It is proved in [16, Theorem S.3.1] and [4, §8.7] that there exists a tempered Borel measurable function $r : \Sigma_0 \to (0, +\infty)$ such that

$$\psi_x : B(0, r(x)) \subset \mathbb{R}^n \to M \quad \text{with} \quad \psi_x(0) = x,$$

is an embedding; moreover

$$\mathrm{dist}_{C^1}(f_x|_{B(0,r(x))}, A(x)) < \epsilon, \tag{51}$$

where $f_x := \psi^{-1}_{f(x)} \circ f \circ \psi_x$ is the representative of f in the given coordinates. In particular,

$$f_x(v, w) = (A^s(x)v + \phi^s(v, w), A^u w + \phi^s(v, w)) \quad \text{with} \quad \|(\phi^s, \phi^u)\|_{C^1(B(0,r(x))} < \epsilon.$$

Let $\sigma_x : [-1, 1]^n \to [-t(x), t(x)]^n$ be the linear rescaling onto the maximal cube contained in $B(0, r(x))$.

Definition 8.1. The rectangle $\mathsf{R}(x)$ is the image of the cube $[-1, 1]^n \subset \mathbb{R}^n$ under

$$e_x := \psi_x \circ \sigma_x : [-1, 1]^n \to M.$$

8.2. **Cylinders and hyperbolic branches.** The crucial feature of a regular neighborhood is that it admits a coordinate system in which the dynamics is essentially uniformly hyperbolic and in particular it defines locally certain approximate *stable* and *unstable* directions which are transversal to each other. Let us fix some $0 < \gamma < 1/2$ and decompose the unit radius cube as a product $I^n = I^s \times I^u$.

Definition 8.2. A *stable admissible manifold* is a graph $\gamma^s = \{e_x(z, \hat{s}(z)) : z \in I^s\}$, where $\hat{s} \in C^1(I^s, I^u)$ is a smooth map with $Lip(\hat{s}) := \sup_{z \in I^s} \|D\hat{s}(z)\| \leq \gamma$.

Admissible manifolds endow $\mathsf{R}(x)$ with a product structure: any given pair of admissible manifolds γ^s and γ^u intersectes transversally at a unique point with an angle bounded from below. Moreover, the map $(\gamma^s, \gamma^u) \to \gamma^s \cap \gamma^u$ so defined satisfies a Lipschitz condition [16, §3.b] and [4, §8]. The transversal structure of the admissible stable and unstable manifolds inside a rectangle R allows us to define the notion of *admissible stable and unstable cylinders*. An *admissible stable cylinder* $S \subseteq \mathsf{R}$ is a subrectangle of R whose boundaries are piecewise smooth sets foliated by segments of admissible stable and unstable manifolds such that the stable manifolds stretch fully across the rectangle R, and similarly, an *admissible unstable cylinder* $U \subseteq \mathsf{R}$ is a subrectangle of R whose boundaries are segments of admissible stable and unstable manifolds such that the unstable manifolds stretch fully across the rectangle R.

The notion of admissible manifold is related to certain cone fields K^s, K^u. For every $z \in \mathsf{R}$ we define $\mathsf{K}^s_z \subset T_zM$ as the image under $De_x(p)$ evaluated at $p(z) = e_x^{-1}(z) \in I^n$, of the cone of width γ 'centered' at $\mathbb{R}^s \oplus \{0\}$, that is, the set of vectors in $\mathbb{R}^n$ making an angle bounded by γ with $\mathbb{R}^s \oplus \{0\}$. We define $\mathsf{K}^u_z \subset T_zM$ likewise considering a cone of width γ 'centered' at $\{0\} \oplus \mathbb{R}^u$. Notice that admissible manifolds are exactly those smooth graph-like submanifolds whose tangent spaces rest inside stable and unstable cones.

We say that a C^1 diffeomorphism $g : S \to U$ between admissible cylinders is *hyperbolic* if it preserves the cone fields K^s and K^u, that is,

$$Dg(z)\mathsf{K}^u_z \subset int\, \mathsf{K}^u_{g(z)}\ \forall\, z \in S \quad \text{and} \quad Dg^{-1}(z)\mathsf{K}^s_z \subset int\, \mathsf{K}^s_{g^{-1}(z)}\ \forall\, z \in U,$$

Definition 8.3. Let R and Q be regular rectangles. If some iterate f^m maps an admissible stable cylinder $S \subset \mathsf{R}$ diffeomorphically and hyperbolically to an admissible unstable cylinder $U \subset \mathsf{Q}$, we shall say that

$$f^m : S \to U$$

is a *hyperbolic branch.*

8.3. **Uniformly hyperbolic Pesin sets.** We now introduce a standard "filtration" of μ almost every point which gives a countable number of nested, uniformly hyperbolic (but not f-invariant) sets, often referred to as "Pesin sets", whose points admit uniform hyperbolic bounds and uniform lower bounds on the sizes of the local stable and unstable manifolds.

For $\chi > 0$ as in (11) above, and every positive integer $\ell > 0$, we define a (possibly empty) compact (not necessarily invariant) set $\Lambda_{\chi,\ell} \subset M$ such that $E^s|_{\Lambda_{\chi,\ell}}$ and $E^u|_{\Lambda_{\chi,\ell}}$ vary continuously with the point $x \in \Lambda_{\chi,\ell}$ and such that

$$\begin{cases} \|Df^n(x)v\| \leq \ell e^{-n\chi}\|v\| & \|Df^{-n}(x)v\| \geq \ell^{-1}e^{n\chi}\|v\|\ \forall\, v \in E^s(x)\ \forall\, n \geq 0 \\ \|Df^{-n}(x)v\| \leq \ell e^{-n\chi}\|v\| & \|Df^{n}(x)v\| \geq \ell^{-1}e^{n\chi}\|v\|\ \forall\, v \in E^s(x)\ \forall\, n \geq 0. \end{cases}$$

Moreover, the angles between the stable and unstable subspaces satisfy

$$\angle(E^s(x), E^u(x)) \geq \ell^{-1}$$

for every $x \in \Lambda_{\chi,\ell}$. As the rate of hyperbolicity of μ is bounded from below by $\chi > 0$ we have

$$\mu(\Lambda_{\chi,\ell}) \to 1 \quad \text{as} \quad \ell \to +\infty.$$

The following result is proved in [16, Theorem S.4.3].

Lemma 8.4. *For every $0 < \alpha < 1$ there exists a $\Lambda = \Lambda_{\chi,\ell}$ with $\mu(\Lambda) \geq 1 - \alpha$ such that $\mathsf{R}(x)$ vary continuously with $x \in \Lambda$, meaning that linear distortion $D(x)$, size of Lyapunov charts $r(x)$ and $x \mapsto e_x \in Embedd(I^s \times I^u, M)$ are continuous functions on Λ.*

We therefore fix some α and let Λ be the set given by Lemma 8.4. Let $r_\Lambda := \min_{x\in\Lambda} r(x) > 0$ be the minimal radius for Lyapunov charts for $x \in \Lambda$ and denote

$$\sigma_x^0 : [-1, 1]^n \to [-t_0, t_0]^n$$

the linear rescaling onto the maximal cube contained in $B(0, r_\Lambda/2)$.

Definition 8.5. Given $0 < h < 1$, we define the *h-reduced rectangle of center x* as the image of the cube $[-h, h]^n \subset [-1, 1]^n$ under the parametrization $e_x^0 := \psi_x \circ \sigma_x^0$, i.e.

$$R^0(x, h) := e_x^0([-h, h]^n).$$

The next proposition is the main technical tool of Pesin's theory to build up Alekseev sets from suitable returns to Pesin sets with a hyperbolic product structure.

8.4. **Markov covers.**

Proposition 8.6. *For every Pesin set Λ there exists constants $0 < h < 1$, $\lambda > 1$, $\kappa > 0$ and a finite covering by reduced rectangles $\{\mathsf{R}_1 = \mathsf{R}^0(p_1, h), \dots, \mathsf{R}_t = \mathsf{R}^0(p_t, h)\}$ such that:*

1. *$\Lambda \subseteq \bigcup_{i=1}^t B(p_i, \kappa)$, and $B(p_i, \kappa) \subset int\, \mathsf{R}_i$;*
2. *$diam(\mathsf{R}_i) \leq \epsilon$ for $i = 1, \cdots, t$;*
3. *if $x \in \Lambda$ returns into $m > 0$ iterates to Λ and $dist(x, p_i) < \kappa$ and $dist(f^m(x), p_j) < \kappa$ then there exists an admissible stable cylinder $S_i \subset \mathsf{R}_i$ containing x and an admissible unstable cylinder $U_j \subset \mathsf{R}_j$ containing $f^m(x)$ such that $f^m : S_i \to U_j$ is a hyperbolic branch with nonlinear rate of expansion bounded from below by $\lambda > 1$, that is:*
$$d_{W'}(f^m(w), f^m(w')) \geq \lambda^m\, d_W(w, w') \quad \forall\ w\,,\ w' \in W \cap S(x)$$
wheredist$_W$ and dist$_{W'}$ is the metric induced by the Lyapunov charts on $W \in U_{\gamma,\rho}(x)$ and $W' = f^m(W \cap S(x))$, respectively.
4. *$diam(f^k(S_i)) \leq \epsilon$, for $0 \leq k \leq m$;*

See [16, Definition S.4.15] and [Theorem S.4.16] [16].

Definition 8.7. A finite family of rectangles $\{\mathsf{R}_1, \cdots, \mathsf{R}_t\}$ satisfying the above properties is called a $(\epsilon, \lambda, \kappa)$-*Markov cover*. We call *kernel of the covering* the family of rectangles $\ker\{\mathcal{R}\} := \{\mathsf{Q}_i\}$ defined by

$$\mathsf{Q}_i := \mathsf{R}^0(p_i, \kappa h). \tag{52}$$

9. Proof of Main Technical Lemma: first step, choosing Ω

Main Technical Lemma follows inmediatly from next lemma taking $\Omega_n = \Omega(\rho_n, s_n, \phi)$, where $\rho_n \downarrow 0^+$ and $s_n \to +\infty$ are suitable sequences.

Lemma 9.1. *Let $\rho, s > 0$, ϕ continuous and μ an ergodic non atomic hyperbolic Borel probability. Then, there exist a hyperbolic basic set*

$$\Omega = \Omega(\rho, s, \phi)$$

such that:

1. *every ergodic measure ν supported on Ω belongs to the basic weak-* open neighborhood $\mathcal{O}(\rho, s)$*

(2) *and the following estimate holds*

$$\frac{\rho \inf \phi}{1+\rho} + \frac{P_\mu(\phi) - 3\rho}{1+\rho} \le P(f|\Omega, \phi) \le P_\mu(\phi) + 2\rho + \rho \sup \phi. \tag{53}$$

We dedicate the rest of this paper to prove Lemma 9.1.

The first step will be to choose Ω. We start with the following

Lemma 9.2. *Let $A \subset M$ be a Borel set with $\mu(A) > 0$. Given $\rho > 0$ and $n > 0$ define*

$$A_{\rho,n} := \{x \in A : x \quad \text{has a return time} \quad R(x) \in [n, (1+\rho)n]\}$$

Then given $0 < \epsilon < 1$ there exists $N > 0$ and a Borel subset $A_\epsilon \subset A$ such that

$$\mu(A_{\rho,n}) \ge (1-\epsilon)\mu(A) \quad \text{for every } n \ge N.$$

Proof. This follows from the ergodicity of μ. Cf. [16]. □

Recall that $B(x, \delta, n) := \{y \in M : \operatorname{dist}(f^j(x), f^j(y)) < \delta \ j = o, \cdots n-1\}$. Let A be a subset. We call a family $\mathcal{B} = \{B(x_i, \delta, n)\}$ a (δ, n)-cover of A if $A \subset \bigcup_i B(x_i, \delta, n)$.

Lemma 9.3. *Let $A \subset \Lambda$ be a compact set with $\mu(A) > 0$. Then there exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ there exists a positive integer $N = N(A, \delta) > 0$ such that every finite (δ, n)-cover $\mathcal{B}$ has at least cardinality two, that is:*

$$\#\mathcal{B} > 1.$$

Lemma 9.4. *For every small $\rho > 0$, $s > 0$ and ϕ continuous there exists:*

a) *a Markov cover $\mathcal{R} = \{\mathsf{R}_\ell\}$ of Pesin set Λ of generic points,*
b) *a positive integer $n > 0$ and*
c) *a family of hyperbolic branches $\{f^{R(x)} : S_x \to U_x;\ x \in E\}$ indexed by a finite set E_0,*

such that:

(1) $\exp(n\rho) \ge \#\mathcal{R}$*;*
(2) $R(x) \in [n, (1+\rho)n]$ *for every* $x \in E_0$*;*
(3) $\#E_\ell > 1$ *for every set* $E_\ell := E_0 \cap \mathsf{R}_\ell$*, the set points of* E_0 *contained in* R_ℓ*;*
(4) *for every rectangle* R_ℓ*, the subfamily of cylinders* $\{S_x; x \in E_\ell\}$ *is disjoint;*
(5) $f^{R(x)} : S_x \to U_x$ *is* (ρ, s) *is quasi-generic (cf. definition 7.4) and the Birkhoff's sums satisfy the following* bounded variation property*:*

$$|S_{R(x)}\phi(y) - S_{R(x)}\phi(z)| < R(x)\rho \quad \text{for every} \quad y, z \in S_x; \tag{54}$$

for every $x \in E_0$*;*

(6) *the partition function* $Z = \sum_{x \in E_0} \exp(S_n\phi(x)$ *is a good estimate for the statistic* $P_\mu(\phi)$*, up to finite precision, more precisely:*

$$\left| \frac{1}{n} \log \left(\sum_{x \in E_0} \exp(S_n\phi(x)) \right) - P_\mu(\phi) \right| < \rho. \tag{55}$$

Proof. We fix

$$\alpha := \mu(\Lambda)/2$$

and choose $\delta = \delta(\rho, s, \phi) > 0$ sufficiently small such that :

- $\operatorname{dist}(x, y) < \delta$ implies

$$|\psi_i(x) - \psi_i(y)| < \rho/2 \qquad \forall\ i \le s \tag{56}$$

$$|\phi(x) - \phi(y)| < \rho \tag{57}$$

which is possible by continuity;

- and

$$\lim_{n\to+\infty}\left|\frac{1}{n}\log\left(\inf_{E}\left\{\sum_{x\in E}\exp S_n\phi(x)\right\}\right)-P_\mu(\phi)\right|<\rho/4, \tag{58}$$

  infimum is taken over all the $(\delta/2, n, \alpha)$-spanning sets.

By Proposition 8.6 there exists a Markov cover $\mathcal{R}=\{\mathsf{R}_\ell\}$ of Λ such that every admissible return $(x, f^{R(x)}(x))$ to its kernel $ker\mathcal{R} := \{\mathsf{Q}_\ell\}$ originates a hyperbolic branch $f^{R(x)}: S_x \to U_x$ subordinated to $\mathcal{R}$ such that

$$\operatorname{diam}(f^j(S_x)) < \delta/4 \quad \text{for every } j = 0, \cdots, R(x)-1. \tag{59}$$

By our choice of constants (56) and (57) this implies that $f^{R(x)}: S_x \to U_x$ is (ρ, s) is (ρ, s)-quasi-generic and Birkhoff's sums satisfy the bounded variation property (54).

Now we choose:

(1) $N_{exp} = N(\exp, \rho, \mathcal{R}) > 0$ such that $\exp(n\rho) \geq \#\mathcal{R}$ for every $n \geq N_1$;
(2) $N_\ell = N(\rho, \mathsf{Q}_\ell)$ such that, for every $n \geq N_\ell$ there exists $\Lambda_\ell \subset \Lambda \cap \mathsf{Q}_\ell$ with

$$\mu(\Lambda_\ell) \geq \mu(\Lambda\cap\mathsf{Q}_\ell)/2,$$

   such that every $x \in \Lambda_\ell$ has an admissible return $f^{R(x)}(x) \in \Lambda_\ell$, with $R(x) \in [n, (1+\rho)n]$; this is a consequence of Lemma 9.2;
(3) $N_\phi = N(\rho, \delta, \alpha, P_\mu(\phi)) > 0$ such that

$$\left|\frac{1}{n}\log\left(\inf_{E}\left\{\sum_{x\in E}\exp S_n\phi(x)\right\}\right)-P_\mu(\phi)\right|<\rho/2, \quad \forall n\geq N_\phi \tag{60}$$

   infimum is taken over all the $(\delta/2, n, \alpha)$-spanning sets; this follows from the definition of limits and (58);
(4) $N_{\mathcal{R}} = N(\Lambda, \mathcal{R}, \delta) > 0$ such that for each Λ_ℓ we have $\min \#\mathcal{B}(n, \delta, \Lambda_\ell) > 1$ for every $n \geq N_{\mathcal{R}}$, where $\mathcal{B}(n, \delta, \Lambda_\ell)$ is the family of $(n, \delta/4)$-covers of Λ_ℓ. This follows from Lemma 9.3.

Fix now

$$n \geq \max_{\ell}\{N_{exp}, N_\ell, N_\phi, N_{\mathcal{R}}\}.$$

and define

$$\Lambda_0 := \bigcup_{\ell}\Lambda_\ell.$$

Clearly, $\mu(\Lambda_0) \geq \alpha$. Then, as every maximal $(\delta/2, n)$-separated subset $E_0 \subset \Lambda_0$ is $(\delta/2, n, \alpha)$-spanning we can choose, for our convenience, a maximal $(\delta/2, n)$-separated subset $E_0 \subset \Lambda_0$ such that:

$$\left|\frac{1}{n}\log\left(\sum_{x\in E_0}\exp S_n\phi(x)\right)-P_\mu(\phi)\right|<\rho.$$

Let us denote $E_\ell := E_0 \cap \mathsf{R}_\ell$. By our choice of n, $\#E_\ell > 1$ for every ℓ. Moreover, every $x \in E_\ell$ originates an admissible return with $R(x) \in [n, (1+\rho)n]$ and therefore, a hyperbolic branch $f^{R(x)}: S_x \to U_x$ in R_ℓ. The family of cylinders $\{\, S_x\,;\, x \in E_\ell \,\}$ so defined is disjoint, since E_0 is $(\delta/2, n)$-separated. Actually, if $S_x \cap S_{x'} \neq \emptyset$ then $\operatorname{dist}(f^j(x), f^j(x')) < \delta/2$ for $j = 0, \cdots, R(x)-1$, by triangular inequality and (59). But this is absurd since $R(x) \geq n$ and E_0 is $(\delta/2, n)$-separated. Therefore, $\{\, f^{R(x)}: S_x \to U_x\,;\, x \in E \,\}$ is the family we were looking for. $\square$

As is inmediate from our choices in previous Lemma, for every rectangle R_ℓ the family $\{\, f^{R(x)} : S_x \to U_x \,;\, x \in E \cap \mathsf{R}_\ell \,\}$ originates a non trivial Alekseev set $\Omega^*_\ell \subset \mathsf{R}_\ell$. Then we are left with the choice of a candidate to approximating set.

Choosing an Aleeksev set:

We choose ℓ maximizing the sums $\sum_{x\in E_\ell} \exp S_n\phi(x)$, that is:

$$\sum_{x\in E_\ell} \exp S_n\phi(x) \geq \sum_{x\in E_{\ell'}} \exp S_n\phi(x) \quad \textit{for every} \quad \ell' \neq \ell. \tag{61}$$

and denote

$$E = E_\ell.$$

Definition 9.5. Let ℓ as defined by (61). We denote $\Omega^*(\rho, s, \phi)$ the Alekseev set defined by the family $\{f^{R(x)} : S_x \to U_x;\ x \in E\}$ and $\Omega(\rho, s, \phi)$ its f-invariant saturate.

In what follows we shall prove that $\Omega(\rho, s, \phi)$ satisfies the claimed properties of Lemma 9.1. From now on, unless notice in contrary, we shall denote $\Omega^* := \Omega^*(\rho, s, \phi)$ and $\Omega := \Omega(\rho, s, \phi)$.

Corollary 9.1. *For every ergodic Borel probability ν supported on Ω it holds*

$$\nu \in \mathcal{O}(3\rho, s)$$

Proof. It is an inmediate consequence of Proposition 7.5 since, by construction, the hyperbolic branches generating Ω^* are (ρ, s) quasi-generic, according to Lemma 9.4. ☐

10. Second step: counting periodic orbits

To estimate the topological pressure of $\Omega(\rho, s, \phi)$ we need to estimate the cardinality of periodic orbits.

Lemma 10.1. *Let Ω be a compact f-invariant, locally maximal, transitive, uniformly hyperbolic set of a C^r $(r \geq 1)$ diffeomorphism and ϕ continuous. Then:*

$$P(\Omega, \phi) = \lim_{N\to+\infty} \frac{1}{N} \log \left(\sum_{x\in Per(N)} \exp S_N\phi \right). \tag{62}$$

To make this computation one has to keep track of the combinatorics of periodic orbits. This is done as follows.

We denote $Per(N)$ the set of periodic points in Ω of prime period N and $E^p = E \times \cdots \times E$ the cartesian product of p-copies of E.

Lemma 10.2. *$f^R|\Omega^*$ is topologically conjugated to the full-shift on $\#E$ symbols.*

Proof. We first observe that, as the stable cylinders S_x, $x \in E$ are disjoint and

$$\Omega^* = \bigcap_{n\mathbb{Z}} (f^R)^{(n)} \left(\bigcup_{x\in E} S_x \right),$$

where $x_n = (f^R)^{(n)}(x)$ is defined inductively as

$$\begin{cases} x_{n+1} = f^{R(x_n)}(x_n) & \text{for } n \geq 0 \\ x_{n-1} = f^{-R(x_n)}(x_n) & \text{for } n \leq 0 \end{cases}$$

setting $x = x_0$. That is,

$$\begin{cases} (f^R)^{(n)}(x) = f^{\sum_{0\le i<n} R(x_i)}(x) & \text{for } n \ge 0 \\ (f^R)^{(n)}(x) = f^{-\sum_{n<i\le 0} R(x_i)}(x) & \text{for } n \le 0 \end{cases}$$

In particular, for every $z \in \Omega^*$ there exists a unique $x \in E$ such that

$$\text{dist}(f^j(x), f^j(z)) < \delta/4 \quad \text{for every} \quad j = 0, \cdots, R(x) - 1.$$

Unicity of $x \in E$ follows since E is part of a $\delta/2$-separated set. Then we associate to every $x \in \Omega^*$ a unique bi-infinite sequence $\{x_n\} \in E^{\mathbb{Z}}$ defined by sucessively $\delta/4$-shadowing of the orbit of z. This shows that $f^R|\Omega^*$ is topologically conjugated to the full-shift on $\#E$ symbols. □

Corollary 10.1. *If $z \in \Omega$ is f-periodic, and assuming without loss of generality that $z \in \Omega^*$, then its succesive returns to Ω^* define an f^R-periodic orbit given by a uniquely defined sequence $[x_0, \cdots, x_{p-1}]$, $p > 1$, $x_k \in E$ such that*

$$x_{k+1} = f^R(x_k) \text{ mod } p$$

and

$$dist(f^{j+\sum_{i<k} R(x_i)}(z), f^j(x_k)) < \delta/4 \quad for \quad j = 0, \cdots, R(x_k) - 1. \tag{63}$$

for $k = 0, \cdots p - 1$.

Remark 10.1. Let $z \in Per(f|\Omega)$ a periodic point. Then according to our previous discussion its prime period $N = N(z)$ is a linear combination of the basic periods $n_1, \cdots, n_{\#E}$ of the branches generating Ω^*, namely, there exists integers $p_i \in \mathbb{Z}^+$, $i = 1, \cdots \#E$ such that

$$N = p_1 n_1 + \cdots p_{\#E} n_{\#E}. \tag{64}$$

Definition 10.3. We say that $N \in \mathbb{Z}^+$ is an admissible period if it satisfies (64) for some sequence of non-negative integers $p_i \in \mathbb{Z}^+$, $i = 1, \cdots \#E$.

Lemma 10.4. *Let $N > 0$ be an admissible period. Then, for every periodic point $z \in Per(N)$ there exists an integer $p > 0$ such that, if $[x_0, \cdots, x_{p-1}] \in E^p$ is the unique sequence of points in E which sucessively $\delta/4$-shadows $\mathcal{O}(z)$ when it cycles around Ω, then*

$$\frac{N}{n(1+\rho)} \le p \le \frac{N}{n}. \tag{65}$$

Proof. Let $N = N(z)$ the prime period of z then

$$N = \sum_{i=0}^{p-1} R(x_i). \tag{66}$$

Using (66) and that $R(x_k) \in [n, (1+\rho)n]$ for every $k = 0, \cdots, p-1$, one concludes that p, the number of cycles around Ω before to return to the initial condition, satisfies the bounds in (65). □

Lemma 10.5. *Let N be an admissible period, $z \in Per(N)$ a periodic point for $f|\Omega$ of prime period N and*

$$[x_0, \cdots, x_{p-1}] \in E^p$$

the encoding sequence defined in Corollary 10.1. Then,

$$\exp\left(S_N(\phi + \rho)(z)\right) \ge \prod_{k=0}^{p-1} \exp\left(S_{R(x_k)}\phi(x_k)\right) \tag{67}$$

and

$$\exp\left(S_N(\phi-\rho)(z)\right) \leq \prod_{i=0}^{p-1} \exp\left(S_{R(x_k)}\phi(x_k)\right). \tag{68}$$

Proof. Recall that the branches originating the Alekseev set Ω^* satisfy

$$\left|\sum_{j=0}^{R(x)-1} \phi(f^j(y)) - \sum_{j=0}^{R(x)-1} \phi(f^j(z))\right| < R(x)\rho \quad \text{for every} \quad y, z \in S_x.$$

Then, we use (66) and (63) to prove

$$\left|\sum_{j=0}^{N-1} \phi(f^j(z)) - \sum_{k=0}^{p-1} \sum_{j=0}^{R(x_k)-1} \phi(f^j(x_k))\right| < N\rho.$$

Therefore

$$S_N(\phi+\rho)(z) \geq \sum_{k=0}^{p-1} \sum_{j=0}^{R(x_k)-1} \phi(f^j(x_k)) \quad \text{and} \quad S_N(\phi-\rho)(z) \leq \sum_{k=0}^{p-1} \sum_{j=0}^{R(x_k)-1} \phi(f^j(x_k)),$$

using again $N = \sum_{k=0}^{p-1} R(x_k)$. Taking the exponential at both sides we get (67) and (68). □

11. Third step: pressure estimates and conclusion

To finish the proof we need to estimate the pressure of the f-invariant saturate of the Alekseev set chosen in previous step.

Definition 11.1. Let $p > 0$ be any positive integer. We denote $\Delta(p)$ the set of admissible periods of periodic orbits in $\Omega(\rho, s, \phi)$, encoded into E^p, more precisely:

$$\Delta(p) := \{N \,:\, \exists\, [x_0, \cdots, x_{p-1}] \in E^p \quad \text{such that} \quad N = \sum_{k=0}^{p-1} R(x_k)\,\}. \tag{69}$$

Lemma 11.2. *For every positive integer $p > 0$ it holds:*

$$\sum_{N\in\Delta(p)} \sum_{z\in Per(N)} \exp(S_N(\phi+\rho)(z)) \geq \left[\sum_{x\in E} \exp(S_{R(x)}\phi(x))\right]^p \tag{70}$$

$$\sum_{N\in\Delta(p)} \sum_{z\in Per(N)} \exp(S_N(\phi-\rho)(z)) \leq \left[\sum_{x\in E} \exp(S_{R(x)}\phi(x))\right]^p \tag{71}$$

Proof. Using inequality (67) in Lemma 10.5, (65) and the identity

$$(a_1 + \cdots + a_n)^m = \sum_{[i_1,\cdots,i_m]} a_{i_1}\cdots a_{i_m} \quad \text{where} \quad [i_1,\cdots,i_m] \in \{1,\cdots,n\}^m$$

we get

$$\begin{aligned}\sum_{N\in\Delta(p)} \sum_{z\in Per(N)} \exp(S_N(\phi+\rho)(z)) \quad &\geq \quad \sum_{[x_1,\cdots,x_p]\in E^p} \prod_{k=0}^{p-1} \exp\left(S_{R(x_k)}\phi(x_k)\right) \\ &= \quad \left[\sum_{x\in E} \exp(S_{R(x)}\phi(x))\right]^p,\end{aligned}$$

thus proving (70). Inequality (71) follows similarly using inequality (68) in Lemma 10.5. □

Proof of Lemma 9.1

We first notice that, as $R(x) \in [n,(1+\rho)n]$ for every $x \in E$ we have

$$\sum_{x\in E} \exp(S_{R(x)}\phi(x)) \geq \sum_{x\in E} \exp(S_n\phi(x)) \times \exp(n\rho \inf \phi) \tag{72}$$

and

$$\sum_{x\in E} \exp(S_{R(x)}\phi(x)) \leq \sum_{x\in E} \exp(S_n\phi(x)) \times \exp(n\rho \sup \phi). \tag{73}$$

By (55) in Lemma 10.4 and by the choice of $E = E_\ell$ as the set which maximizes the sums $\sum_{x\in E_\ell} \exp(S_n\phi(x))$ as is (61), we have

$$\#\mathcal{R} \sum_{x\in E} \exp(S_n\phi(x)) \geq \sum_{x\in E_0} \exp(S_n\phi(x)) > \exp(n[P_\mu(\phi) - \rho])$$

thus giving

$$\sum_{x\in E} \exp(S_n\phi(x)) \geq \exp(n[P_\mu(\phi) - 2\rho]), \tag{74}$$

since $\exp(n\rho) > \#\mathcal{R}$, by (1) in Lemma 9.4. On the other hand, as $E \subset E_0$

$$\sum_{x\in E} \exp(S_n\phi(x)) \leq \sum_{x\in E_0} \exp(S_n\phi(x)) < \exp(n[P_\mu(\phi) + \rho]). \tag{75}$$

Therefore, substituting (74) into (72) and recalling (70) we get the lower bound

$$\sum_{N\in\Delta(p)} \sum_{z\in Per(N)} \exp(S_N(\phi+\rho)(z)) \geq [\exp(n[P_\mu(\phi) - 2\rho]) \times \exp(n\rho \inf \phi)]^p \tag{76}$$

Similarly, introducing (75) into (73) and recalling (71) we get

$$\sum_{N\in\Delta(p)} \sum_{z\in Per(N)} \exp(S_N(\phi-\rho)(z)) \leq [\exp(n[P_\mu(\phi) + \rho]) \times \exp(n\rho \sup \phi)]^p \tag{77}$$

On the other hand, as $R(x) \in [n,(1+\rho)]$ we have

$$np \leq N \leq n(1+\rho)p \quad \text{for every} \quad N \in \Delta(p). \tag{78}$$

Therefore,

$$1 \leq \#\Delta(p) \leq n(1+\rho)p - np = np\rho \tag{79}$$

Hence, maximizing the sums $\sum_{z\in Per(N)} \exp(S_N(\phi+\rho)(z))$ over the set of admissible periods $N \in \Delta(p)$ we get

$$\#\Delta(p) \sum_{z\in Per(N_p)} \exp(S_{N_p}(\phi+\rho)(z)) \geq [\exp(n[P_\mu(\phi) - 2\rho]) \times \exp(n\rho \inf \phi)]^{\frac{N_p}{(1+\rho)n}},$$

for a suitable admissible period $N_p \in \Delta(p)$, where we have used (65) to bound from below $p > 0$ in terms of N_p. Then by (79) we get

$$np\rho \times \sum_{z\in Per(N_p)} \exp(S_{N_p}(\phi+\rho)(z)) \geq [\exp(n[P_\mu(\phi) - 2\rho]) \times \exp(n\rho \inf \phi)]^{\frac{N_p}{(1+\rho)n}}. \tag{80}$$

Similarly, minimizing the sums $\sum_{z\in Per(N)}\exp(S_N(\phi-\rho)(z))$ over the set of admissible periods $N\in\Delta(p)$, we get from (77), (65) and (79),

$$\sum_{z\in Per(N_p)}\exp(S_{N_p}(\phi-\rho)(z))\leq[\exp(n[P_\mu(\phi)+\rho])\times\exp(n\rho\sup\phi)]^{\frac{N_p}{n}}, \tag{81}$$

for a suitable $N_p\in[np,n(1+\rho)p]$. Then, by passing to the limit letting $p\to+\infty$, we get a sequence $N_p\to+\infty$ of admissible periods, such that, when taking logarithms and dividing by N_p in (80) and (81) we get, by Lemma 10.1,

$$\begin{aligned} P(f|\Omega,\phi+\rho) &= \lim_{p\to+\infty}\frac{\log(np\rho)}{N_p}+\lim_{p\to+\infty}\frac{1}{N_p}\log\sum_{z\in Per(N_p)}\exp(S_{N_p}(\phi+\rho)(z)) \\ &\geq \frac{P_\mu(\phi)-2\rho}{1+\rho}+\frac{\rho\inf\phi}{1+\rho}, \end{aligned}$$

and

$$\begin{aligned} P(f|\Omega,\phi-\rho) &= \lim_{p\to+\infty}\frac{1}{N_p}\log\sum_{z\in Per(N_p)}\exp(S_{N_p}(\phi-\rho)(z)) \\ &\leq P_\mu(\phi)+\rho+\rho\sup\phi, \end{aligned}$$

where we have used $N_p\geq np$ so that

$$\lim_{p\to+\infty}\frac{\log(np\rho)}{N_p}=0.$$

This proves estimative (53) at Lemma 9.1, after a straightforward calculation, using $P(\phi+c)=P(\phi)$ ([Theorem 2.1, (vii)][32]). **QED**

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